FAST ORTHOGONAL APPROXIMATIONS OF SAMPLED SINUSOIDS AND BANDLIMITED SIGNALS

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ABSTRACT

In this paper, we provide a dictionary for representing the discrete vector one obtains when collecting a finite set of uniform samples from a baseband analog signal. Like the discrete prolate spheroidal sequences (DPSS's), the proposed orthogonal basis compactly captures most of the energy in oversampled bandlimited signals. The complexity of computing the representation of a signal using the proposed dictionary is comparable to the FFT, which is much less than that involving the DPSS basis. We also give non-asymptotic results to guarantee that the proposed basis not only provides a very high degree of approximation accuracy in an MSE sense for bandlimited sample vectors, but also that it can provide high-quality approximations of all sampled sinusoids within the band of interest.

***Index Terms*—** Discrete prolate spheroidal sequences, fast Fourier transform, bandlimited signal, signal approximation

1. INTRODUCTION

The Nyquist-Shannon sampling theorem guarantees that real world signals that are bandlimited (or can be made bandlimited by filtering) can be replaced by a discrete sequence of their samples without the loss of any information and processed digitally. The discrete Fourier transform (DFT) for digital signals and has been widely used for many applications in engineering, mathematics, and science thanks to the fast Fourier tranform (FFT), an efficient algorithm for computing the DFT.

However, due to the fact that windowing in time domain will spread out the spectrum in the frequency domain, the DFT suffers from frequency leakage when used to represent a finite-length vector arising from a bandlimited signal with narrowband spectrum, or ever a pure sinusoid. Such a problem can be mitigated to some degree by applying a windowing function in the sampling system. Alternatively, one can compactly represent the signals using a basis of timelimited discrete prolate spheroidal sequences (DPSS's). DPSS's, first introduced by Slepian in 1978 [1], are a collection of orthogonal bandlimited sequences that are most concentrated in time to a given index range. When limited in the time domain, they provide a compact (and again orthogonal) representation for sampled bandlimited signals. Owing to their concentration in the time and frequency domains, the DPSS's have been successfully used in numerous signal processing applications such as time-variant channel estimation [2, 3], super-resolution [4], mitigating wall cluter and detecting non-point targets in through-the-wall radar imaging [5–7], signal recovery from compressive measurements [8, 9], and so on.

Unlike the DFT which can be computed efficiently with FFT, there exists no algorithm that can efficiently compute the DPSS representation for a very large signal. Recently, we proposed [10] a fast method for computing approximate projections onto the leading DPSS vectors and compressing a signal to the corresponding low dimension. In this paper, we illustrate an alternative orthonormal basis that compactly captures most of the energy in sampled bandlimited signals, and the representation for an arbitrary vector in this basis can be computed efficiently. Moreover, one of the main contributions of this paper is to confirm that such an orthonormal basis not only provides a very high degree of approximation accuracy in a mean squared error (MSE) sense for baseband sample vectors, but also that it can provide high-quality approximations for all sample vectors of sinusoids with frequencies in the band of interest. The remainder of this paper is organized as follows. Section 2 provides some important background information on DPSS's. We state our main results in Section 3. In Section 4, we present some experiments to illustrate the effectiveness of our proposed approximations.

2. DPSS BASES

To begin, we briefly review some important definitions and properties of DPSS's.

For any $W \in (0, \frac{1}{2})$, let $\mathcal{B}_W : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$ denote a bandlimiting operator that bandlimits the discrete-time Fourier transform (DTFT) of a discrete-time signal to the frequency range $[-W, W]$ (and returns the corresponding signal in the time domain). In addition, for any $N \in \mathbb{N}$, let

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$\mathcal{T}_N : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$ denote the timelimiting operator that zeros out all entries outside the index range $\{0, 1, \ldots, N-1\}$.

Definition 1. *(DPSS's [1]) Given $W \in (0, \frac{1}{2})$ and $N \in \mathbb{N}$, the Discrete Prolate Spheroidal Sequences (DPSS's) $\{s_{N,W}^{(0)}, s_{N,W}^{(1)}, \ldots, s_{N,W}^{(N-1)}\}$ are real-valued discrete-time sequences that satisfy $\mathcal{B}_W(\mathcal{T}_N(s_{N,W}^{(l)})) = \lambda_{N,W}^{(l)} s_{N,W}^{(l)}$ for all $l \in \{0, \ldots, N-1\}$. Here $\lambda_{N,W}^{(0)}, \ldots, \lambda_{N,W}^{(N-1)}$ are the eigenvalues of the operator $\mathcal{B}_{[-W,W]}\mathcal{T}_N$ with order $1 > \lambda_{N,W}^{(0)} > \lambda_{N,W}^{(1)} > \cdots > \lambda_{N,W}^{(N-1)} > 0$.*

Definition 2. *(DPSS vectors [1]) Given $W \in (0, \frac{1}{2})$ and $N \in \mathbb{N}$, the DPSS vectors $\boldsymbol{s}_{N,W}^{(0)}\ \boldsymbol{s}_{N,W}^{(1)}, \ldots,\ \boldsymbol{s}_{N,W}^{(N-1)} \in \mathbb{R}^N$ are defined by limiting the DPSS's to the index range $\{0, 1, \ldots, N-1\}$, i.e.,*

$$\boldsymbol{s}_{N,W}^{(l)}[n] = s_{N,W}^{(l)}[n]$$

and satisfy

$$\boldsymbol{B}_{N,W}\boldsymbol{s}_{N,W}^{(l)} = \lambda_{N,W}^{(l)}\boldsymbol{s}_{N,W}^{(l)},$$

where $\boldsymbol{B}_{N,W} \in \mathbb{C}^{N\times N}$ is the prolate matrix with elements

$$\boldsymbol{B}_{N,W}[m,n] = \frac{\sin(2\pi W(m-n))}{\pi(m-n)}.$$

Let $\boldsymbol{S}_{N,W}$ denote an $N \times N$ matrix whose l-th column is the DPSS vector $\boldsymbol{s}_{N,W}^{(l)}$ for all $l = 0, \ldots, N-1$ and $\boldsymbol{\Lambda}_{N,W}$ be an $N \times N$ diagonal matrix with diagonal entries being the DPSS eigenvalues $\lambda_{N,W}^{(0)}, \ldots, \lambda_{N,W}^{(N-1)}$. The prolate matrix $\boldsymbol{B}_{N,W}$ can be factorized as

$$\boldsymbol{B}_{N,W} = \boldsymbol{S}_{N,W}\boldsymbol{\Lambda}_{N,W}\boldsymbol{S}_{N,W}^*,$$

which is an eigendecompostion of $\boldsymbol{B}_{N,W}$. Here $\boldsymbol{A}^*$ represents the conjugate transpose of $\boldsymbol{A}$. The DPSS's are orthogonal on $\mathbb{Z}$ and on $\{0, 1, \ldots, N-1\}$, and they are normalized so that

$$\langle \mathcal{T}_N(s_{N,W}^{(k)}), \mathcal{T}_N(s_{N,W}^{(l)})\rangle = \begin{cases} 1, & k = l, \\ 0, & k \neq l. \end{cases}$$

Consequently, it can be shown [1] that $\|s_{N,W}^{(l)}\|_2^2 = \frac{1}{\lambda_{N,W}^{(l)}}$. Thus, when $\lambda_{N,W}^{(l)}$ is close to 1, the corresponding DPSS vector $\boldsymbol{s}_{N,W}^{(l)}$ has energy mostly concentrated in the frequency range $[-W, W]$. On the other hand when $\lambda_{N,W}^{(l)}$ is close to 0, the corresponding DPSS vector $\boldsymbol{s}_{N,W}^{(l)}$ has most of its energy outside the frequency range $[-W, W]$. These properties, along with the following result on the distribution of the eigenvalues $\lambda_{N,W}^{(l)}$, make the DPSS's a suitable basis to provide a compact representation for sampled bandlimited signals.

Theorem 1. *(Concentration of the spectrum [1, 9–11].) For any $W \in (0, \frac{1}{2})$, $N \in \mathbb{N}$, and $\epsilon \in (0, \frac{1}{2})$, we have*

$$\lambda_{N,W}^{(\lfloor 2NW \rfloor - 1)} \geq \frac{1}{2} \geq \lambda_{N,W}^{(\lceil 2NW \rceil)}$$

and

$$\#\{\epsilon \leq \lambda_{N,W}^{(l)} \leq 1-\epsilon\} \leq 2C_N \log\left(\frac{15}{\epsilon}\right),$$

where $C_N = \frac{4}{\pi^2}\log(8N) + 6$.

Here $\lfloor a \rfloor$ denotes the largest integer that is not greater than a and $\lceil a \rceil$ denotes the smallest integer that is not smaller than a. Theorem 1 implies that the first $\approx 2NW$ eigenvalues tend to cluster very close to 1, while the remaining eigenvalues tend to cluster very close to 0, after a narrow transition of width $O(\log(N)\log(\frac{1}{\epsilon}))$.

Define

$$\boldsymbol{e}_f := [e^{j2\pi f0}\ e^{j2\pi f1}\ \cdots\ e^{j2\pi f(N-1)}]^{\mathrm{T}} \in \mathbb{C}^N$$

for all $f \in [-\frac{1}{2}, \frac{1}{2}]$ as the sampled exponentials, where T represents the transpose operator. For any integer $K \in \{1, 2, \ldots, N\}$, let $\boldsymbol{S}_K := [\boldsymbol{S}_{N,W}]_K$ denote the $N \times K$ matrix formed by taking the first K DPSS vectors. Note that for any orthonormal matrix $\boldsymbol{Q} \in \mathbb{C}^{N\times K}$,

$$\begin{aligned} &\frac{1}{2W}\int_{-W}^{W} \|\boldsymbol{e}_f - \boldsymbol{Q}\boldsymbol{Q}^*\boldsymbol{e}_f\|_2^2\, df \\ &= \frac{1}{2W}\operatorname{trace}(\boldsymbol{B}_{N,W} - \boldsymbol{Q}\boldsymbol{Q}^*\boldsymbol{B}_{N,W}). \end{aligned} \tag{1}$$

This implies that $\boldsymbol{S}_K$ is the best basis of K columns to represent all sampled sinusoids $\{\boldsymbol{e}_f\}_{f\in[-W,W]}$ in the least-squares sense. Formally,

$$\frac{1}{2W}\int_{-W}^{W} ||\boldsymbol{e}_f - \boldsymbol{S}_K\boldsymbol{S}_K^*\boldsymbol{e}_f||_2^2 df = \frac{1}{2W}\sum_{l=K}^{N-1}\lambda_{N,W}^{(l)},$$

whereas for each $f \in [-W, W]$, $\|\boldsymbol{e}_f\|_2^2 = N$. It follows from Theorem 1 that $\boldsymbol{S}_K$ provides very accurate approximations (in an MSE sense) for all sampled sinusoids $\{\boldsymbol{e}_f\}_{f\in[-W,W]}$ if one chooses K slightly larger than $2NW$.

We note that any representation guarantee for sampled sinusoids $\{\boldsymbol{e}_f\}_{f\in[-W,W]}$ can also be used for finite-length sample vectors arising from sampling random bandlimited baseband signals. Suppose x is a continuous-time, zero-mean, wide sense stationary random process with power spectrum

$$P_x(F) = \begin{cases} \frac{1}{B_{\text{band}}}, & F \in [-\frac{B_{\text{band}}}{2}, \frac{B_{\text{band}}}{2}], \\ 0, & \text{otherwise}. \end{cases}$$

Let $\boldsymbol{x} = [x(0)\ x(T_s)\ \cdots\ x((N-1)T_s)]^T \in \mathbb{C}^N$ denote a finite vector of samples acquired from $x(t)$ with a sampling interval of $T_s \leq 1/B_{\text{band}}$. Let $f_c = F_c T_s$ and $W = \frac{B_{\text{band}} T_s}{2}$. We have [9]

$$\mathrm{E}\left[||\boldsymbol{x} - \boldsymbol{Q}\boldsymbol{Q}^*\boldsymbol{x}||_2^2\right] = \frac{1}{2W}\int_{-W}^{W} ||\boldsymbol{e}_f - \boldsymbol{Q}\boldsymbol{Q}^*\boldsymbol{e}_f||_2^2 df. \quad (2)$$

Let $\boldsymbol{F}_{N,W}$ denote the partial normalized DFT matrix with the lowest $2\lfloor NW \rfloor + 1$ frequency DFT vectors of length N, i.e.,

$$\boldsymbol{F}_{N,W} = \left[\begin{array}{ccc} \frac{1}{\sqrt{N}}\boldsymbol{e}_{-\frac{\lfloor NW \rfloor}{N}} & \cdots & \frac{1}{\sqrt{N}}\boldsymbol{e}_{\frac{\lfloor NW \rfloor}{N}} \end{array}\right].$$

It follows that $\boldsymbol{F}_{N,W}\boldsymbol{F}_{N,W}^*$ is an orthogonal projector onto the column space of $\boldsymbol{F}_{N,W}$. The following result states that the difference between the prolate matrix $\boldsymbol{B}_{N,W}$ and $\boldsymbol{F}_{N,W}\boldsymbol{F}_{N,W}^*$ is effectively low rank.

Theorem 2. *[10] Let $N \in \mathbb{N}$ and $W \in (0, \frac{1}{2})$ be given. Then for any $\epsilon \in (0, \frac{1}{2})$, there exist $N \times N$ matrices $\boldsymbol{L}$ and $\boldsymbol{E}$ such that*

$$\boldsymbol{B}_{N,W} = \boldsymbol{F}_{N,W}\boldsymbol{F}_{N,W}^* + \boldsymbol{L} + \boldsymbol{E},$$

where

$$\operatorname{rank}(\boldsymbol{L}) \leq C_N \log\left(\frac{15}{\epsilon}\right), \quad \|\boldsymbol{E}\| \leq \epsilon.$$

Here C_N is the constant specified in Theorem 1.

This result is a key factor in fast computing an approximate compression onto the Slepian basis in [10] and will play an important role in the following computation of fast orthogonal approximations of sampled sinusoids and bandlimited signals.

3. FAST ORTHOGONAL APPROXIMATIONS

In [10], we demonstrated a fast method to approximately project an arbitrary vector onto the subspace spanned by the first slightly more than $2NW$ eigenvectors of $\boldsymbol{B}_{N,W}$ by utilizing the fact that the difference between $\boldsymbol{B}_{N,W}$ and $\boldsymbol{F}_{N,W}\boldsymbol{F}_{N,W}^*$ approximately has a rank of $O(\log N)$ (see Theorem 2). Note that, in [10], the approximate projection is not a true projection onto any subspace. Here, we exhibit a subspace that captures most of the energy in the sampled sinusoids within the band of interest, and this subspace has an orthogonal projector that can be applied efficiently to an arbitrary vector.

By utilizing the result that $\boldsymbol{B}_{N,W} - \boldsymbol{F}_{N,W}\boldsymbol{F}_{N,W}^*$ is approximately low rank and also that $\boldsymbol{F}_{N,W}$ can be applied to a vector efficiently with the FFT, we build our subspace with an orthonormal basis of the form $\boldsymbol{Q} = \left[\boldsymbol{F}_{N,W} \quad \boldsymbol{Q}'\right]$, where $\boldsymbol{Q}' \in \mathbb{C}^{N \times R}$ for some R that we can choose as desired. Let $\overline{\boldsymbol{F}}_{N,W}$ denote the $N \times (N - 2\lfloor NW \rfloor - 1)$ matrix with the highest frequency $N - 2\lfloor NW \rfloor - 1$ DFT vectors of length N. Thus $\boldsymbol{F}_N := \left[\boldsymbol{F}_{N,W} \quad \overline{\boldsymbol{F}}_{N,W}\right]$ is the normalized DFT matrix. Since $\boldsymbol{Q}'$ must be orthogonal to $\boldsymbol{F}_{N,W}$ and the columns of $\boldsymbol{Q}'$ must be orthonormal, we can represent $\boldsymbol{Q}'$ by $\boldsymbol{Q}' = \overline{\boldsymbol{F}}_{N,W}\boldsymbol{V}$, where $\boldsymbol{V} \in \mathbb{C}^{(N-2\lfloor NW \rfloor -1)\times R}$ is orthonormal (one can verify that $\boldsymbol{F}_{N,W}^*\boldsymbol{Q}' = \boldsymbol{0}$ and $(\boldsymbol{Q}')^*\boldsymbol{Q}' = \mathbf{I}$).

Plugging $\boldsymbol{Q} = \left[\boldsymbol{F}_{N,W} \quad \overline{\boldsymbol{F}}_{N,W}\boldsymbol{V}\right]$ into (1) yields

$$\int_{-W}^{W} \|\boldsymbol{e}_f - \boldsymbol{Q}\boldsymbol{Q}^*\boldsymbol{e}_f\|_2^2\, df$$
$$= \operatorname{trace}\left(\overline{\boldsymbol{F}}_{N,W}^*\boldsymbol{B}_{N,W}\overline{\boldsymbol{F}}_{N,W} - \boldsymbol{V}\boldsymbol{V}^*\overline{\boldsymbol{F}}_{N,W}^*\boldsymbol{B}_{N,W}\overline{\boldsymbol{F}}_{N,W}\right)$$

which suggests that setting $\boldsymbol{V}$ equal to the R dominant left singular vectors of $\overline{\boldsymbol{F}}_{N,W}^*\boldsymbol{B}_{N,W}$ (or $\overline{\boldsymbol{F}}_{N,W}^*\boldsymbol{B}_{N,W}\overline{\boldsymbol{F}}_{N,W}$) results in a relatively small representation residual in the right hand of the above equation as long as $\overline{\boldsymbol{F}}_{N,W}^*\boldsymbol{B}_{N,W}$ has an effective rank of R. The following result provides a formal guarantee on this.

Theorem 3. *(Average representation error) Fix $W \in (0, \frac{1}{2})$ and $N \in \mathbb{N}$. Let $\boldsymbol{V} \in \mathbb{C}^{(N-2\lfloor NW \rfloor -1)\times R}$ contain the R dominant left singular vectors of $\overline{\boldsymbol{F}}_{N,W}^*\boldsymbol{B}_{N,W}$. Then for any $\epsilon \in (0, \frac{1}{2})$, the orthobasis $\boldsymbol{Q} = \left[\boldsymbol{F}_{N,W} \quad \overline{\boldsymbol{F}}_{N,W}\boldsymbol{V}\right]$ satisfies*

$$\frac{1}{2W}\int_{-W}^{W} \frac{\|\boldsymbol{e}_f - \boldsymbol{Q}\boldsymbol{Q}^*\boldsymbol{e}_f\|_2^2}{\|\boldsymbol{e}_f\|_2^2} df \leq \epsilon$$

with

$$R = \max\left\{\left\lceil C_N \log\left(\frac{15C_N}{2NW\epsilon}\right)\right\rceil, 0\right\}.$$

Here C_N is the constant specified in Theorem 1.

A similar approximation guarantee holds for sampled vectors arising from sampling random bandlimited signals by using (2). We note that we are not guaranteed that $\|\boldsymbol{Q}\boldsymbol{Q}^* - \boldsymbol{S}_K\boldsymbol{S}_K^*\|$ is small since in general $\|\boldsymbol{Q}\boldsymbol{Q}^* - \boldsymbol{S}_K\boldsymbol{S}_K^*\| = 1$ if $\boldsymbol{Q}$ and $\boldsymbol{S}_K$ have a different number of columns. However, we are guaranteed that the subspace spanned by the columns of $\boldsymbol{S}_K$ is approximately within the column space of $\boldsymbol{Q}$ by the following result.

Theorem 4. *(Representation guarantee for DPSS vectors) Fix $N \in \mathbb{N}$ and $W \in (0, \frac{1}{2})$. Let $\boldsymbol{V} \in \mathbb{C}^{(N-2\lfloor NW \rfloor -1)\times R}$ be the R dominant left singular vectors of $\overline{\boldsymbol{F}}_{N,W}^*\boldsymbol{B}_{N,W}$. For any $\epsilon \in (0, \frac{1}{2})$, fix K to be such that $\lambda_{N,W}^{(K-1)} \geq \epsilon$. Then the orthobasis $\boldsymbol{Q} = \left[\boldsymbol{F}_{N,W} \quad \overline{\boldsymbol{F}}_{N,W}\boldsymbol{V}\right]$ satisfies*

$$\|\boldsymbol{S}_K\boldsymbol{S}_K^* - \boldsymbol{Q}\boldsymbol{Q}^*\boldsymbol{S}_K\boldsymbol{S}_K^*\|^2 \leq \epsilon$$
$$\|\boldsymbol{s}_{N,W}^{(l)} - \boldsymbol{Q}\boldsymbol{Q}^*\boldsymbol{s}_{N,W}^{(l)}\| \leq \epsilon$$

for all $l = 0, 1, \ldots, K-1$ with $R = \lceil C_N \log(15/\epsilon)\rceil$. Here C_N is the constant specified in Theorem 1.

Note that the bound on $\|\boldsymbol{S}_K\boldsymbol{S}_K^* - \boldsymbol{Q}\boldsymbol{Q}^*\boldsymbol{S}_K\boldsymbol{S}_K^*\|$ is useful since for any vector $\boldsymbol{a} \in \mathbb{C}^N$

$$\begin{aligned}
&\|\boldsymbol{a} - \boldsymbol{Q}\boldsymbol{Q}^*\boldsymbol{a}\| \\
&\leq \|\boldsymbol{a} - \boldsymbol{Q}\boldsymbol{Q}^*\boldsymbol{S}_K\boldsymbol{S}_K^*\boldsymbol{a}\| \\
&\leq \|\boldsymbol{a} - \boldsymbol{S}_K\boldsymbol{S}_K^*\boldsymbol{a}\| + \|\boldsymbol{S}_K\boldsymbol{S}_K^* - \boldsymbol{Q}\boldsymbol{Q}^*\boldsymbol{S}_K\boldsymbol{S}_K^*\|\|\boldsymbol{a}\| \\
&\leq \|\boldsymbol{a} - \boldsymbol{S}_K\boldsymbol{S}_K^*\boldsymbol{a}\| + \sqrt{\epsilon}\|\boldsymbol{a}\|,
\end{aligned}$$

which implies any representation guarantee for $\boldsymbol{S}_K$ can be utilized for $\boldsymbol{Q}$.

In [11], we rigorously show that every discrete-time sinusoid with a frequency $f \in [-W, W]$ is well-approximated by the DPSS basis $\boldsymbol{S}_K$ with K slightly larger than $2NW$. The proof is based on an asymptotic result on the DTFT of the DPSS basis functions (which are known as discrete prolate spheroidal wave functions (DPSWF's)) and the result is thus asymptotic. Here we use a different approach to obtain a non-asymptotic guarantee for approximating every discrete-time sinusoid with a frequency $f \in [-W, W]$. Noting that $\|\boldsymbol{e}_f - \boldsymbol{Q}\boldsymbol{Q}^*\boldsymbol{e}_f\|_2^2$ is differentiable everywhere, we first show that its derivative is bounded above by $2\pi N^2$. Then by utilizing the previous result on $\int_{-W}^{W} \|\boldsymbol{e}_f - \boldsymbol{Q}\boldsymbol{Q}^*\boldsymbol{e}_f\|_2^2\, df$, we obtain a similar bound on $\|\boldsymbol{e}_f - \boldsymbol{Q}\boldsymbol{Q}^*\boldsymbol{e}_f\|_2^2$.

Theorem 5. *(Representation guarantee for pure sinusoids) Fix $N \in \mathbb{N}$ and $W \in (0, \frac{1}{2})$. Let $\boldsymbol{V} \in \mathbb{C}^{(N-2\lfloor NW \rfloor -1)\times R}$ be the R dominant left singular vectors of $\overline{\boldsymbol{F}}_{N,W}^* \boldsymbol{B}_{N,W}$. Then for any $\epsilon \in (0, \frac{1}{2})$, the orthobasis $\boldsymbol{Q} = \begin{bmatrix}\boldsymbol{F}_{N,W} & \overline{\boldsymbol{F}}_{N,W}\boldsymbol{V}\end{bmatrix}$ satisfies*

$$\frac{\|\boldsymbol{e}_f - \boldsymbol{Q}\boldsymbol{Q}^*\boldsymbol{e}_f\|_2^2}{\|\boldsymbol{e}_f\|_2^2} \le \epsilon$$

for all $f \in [-W, W]$ with

$$R = \max\left\{ C_N \log\left(\frac{60\pi C_N}{\epsilon^2}\right), C_N \log\left(\frac{15 C_N}{NW\epsilon}\right)\right\} + 1.$$

Here C_N is the constant specified in Theorem 1.

Finally, we remark that for $\boldsymbol{Q} = \begin{bmatrix}\boldsymbol{F}_{N,W} & \overline{\boldsymbol{F}}_{N,W}\boldsymbol{V}\end{bmatrix}$ with $\boldsymbol{V} \in \mathbb{C}^{(N-2\lfloor NW \rfloor -1)\times R}$, both $\boldsymbol{Q}$ and $\boldsymbol{Q}^*$ can be applied to a vector in $O(N\log N + NR)$. As an example, for any $\boldsymbol{a} \in \mathbb{C}^N$, $\widetilde{\boldsymbol{a}} = \begin{bmatrix}\boldsymbol{F}_{N,W} & \overline{\boldsymbol{F}}_{N,W}\end{bmatrix}^{\mathrm{H}} \boldsymbol{a}$ can be efficiently computed by the FFT with complexity $O(N \log N)$. Then $\boldsymbol{V}^*\widetilde{\boldsymbol{a}}_2$ can be computed via conventional matrix-vector multiplication with complexity $O(NR)$, where $\widetilde{\boldsymbol{a}}_2$ is the sub-vector obtained by taking the last $N - 2\lfloor NW \rfloor - 1$ entries of $\widetilde{\boldsymbol{a}}_2$. Thus the total computational complexity for $\boldsymbol{Q}^*\boldsymbol{a}$ is $O(N\log N + NR)$. We note that R is in the order of at most $\log N \log(\frac{\log N}{W\epsilon^2})$ for Theorems 3-5.

4. SIMULATIONS

In this section, we present some experiments to illustrate the effectiveness of our proposed fast approximation algorithm ROAST (which is short for Rapid Orthogonal Approximate Slepian Transform).

For comparison, we also compute the projection onto the column space of $\boldsymbol{F}_{N,W+\frac{R}{2N}}$ which is the $N \times (2\lfloor NW \rfloor + 1 + R)$ DFT matrix with frequencies in $[-W - \frac{R}{2N}, W + \frac{R}{2N}]$. Such a projection is simply denoted by Sub-DFT. Note that the dimension of the column space of $\boldsymbol{F}_{N,W+\frac{R}{2N}}$ is $2\lfloor NW \rfloor + 1 + R$ and is equal to the dimension of the column space of $\boldsymbol{Q}$.

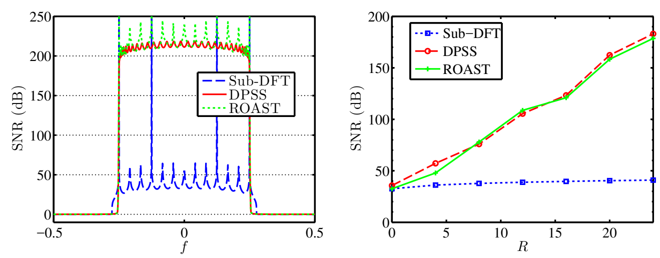
Fig. 1(a) shows the ability of the different projections to capture a given sinusoid in terms of

$$\mathrm{SNR} = 20\log_{10}\left(\frac{\|\boldsymbol{e}_f\|_2}{\|\boldsymbol{e}_f - \widehat{\boldsymbol{e}}_f\|_2}\right)\mathrm{dB},$$

where $\widehat{\boldsymbol{e}}_f$ is the resulting projection of $\boldsymbol{e}_f$ by the above mentioned methods.

Also, we generate a sampled bandlimited signal $\boldsymbol{x}$ by adding 5000 complex exponentials with frequencies selected uniformly at random within the frequency band $[-W, W]$. Fig. 1(b) shows the ability of the different projections to capture sampled bandlimited signals in terms of SNR.

Finally, Fig. 2 plots SNR as a function of dimension N and the relationship between the run time and N for the three projection methods. In this experiment, we fix $R = 4\log(N)$. As observed, the running time of DPSS has a quadratic increase, while ROAST is nearly as fast as the DFT, but with much better approximation performance.



(a) (b)

Fig. 1. (a) SNR captured by different projections for pure sinusoids $\boldsymbol{e}_f$ with $R = 4\log(N)$; (b) SNR captured by different projections for a sampled bandlimited signal $\boldsymbol{x}$ with R ranging from 0 to $4\log(N)$. Here $N = 1024$, $W = \frac{1}{4}$.

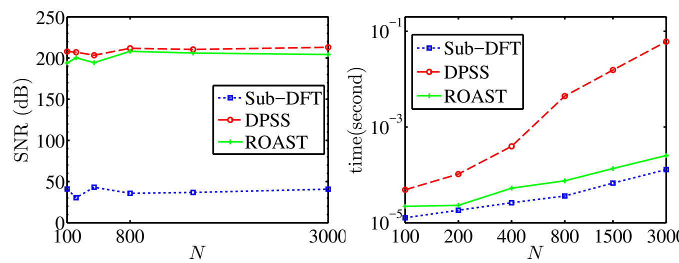


Fig. 2. Comparison of different projections for a sampled bandlimited signal $\boldsymbol{x}$. Left: SNR as a function of N. Right: Computation time as a function of N in $\log_{10}$ scale. In all plots, $W = \frac{1}{4}$ and $R = 4\log(N)$.

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