

FAST COMPUTATIONS FOR APPROXIMATION AND COMPRESSION IN SLEPIAN SPACES

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ABSTRACT

The discrete prolate spheroidal sequences (DPSS's) provide an efficient representation for signals that are perfectly time-limited and nearly bandlimited. Unfortunately, because of the high computational complexity of projecting onto the DPSS basis – also known as the *Slepian basis* – this representation is often overlooked in favor of the fast Fourier transform (FFT). In this paper, we show that there exist fast constructions for computing approximate projections onto the leading Slepian basis elements. The complexity of the resulting algorithms is comparable to the FFT, and scales favorably as the quality of the desired approximation is increased. We demonstrate how these algorithms allow us to efficiently compute the solution to certain least-squares problems that arise in signal processing.

Index Terms— Slepian basis, discrete prolate spheroidal sequences, fast Fourier transform, signal approximation and compression

1. INTRODUCTION

The fact that many real-world signals are exactly or approximately *bandlimited*—or can be made bandlimited via low-pass filtering—is a key enabler of the Digital Signal Processing (DSP) revolution. Thanks to the Shannon-Nyquist sampling theorem, broad classes of signals (namely, bandlimited ones) can be sampled without loss of information and processed digitally on a computer. Thanks also to the fast Fourier transform (FFT), a fast algorithm for computing a signal's discrete Fourier transform (DFT), many of these computations can be performed quickly even for very large signals.

Unfortunately, the DFT suffers certain shortcomings when used to represent finite-length sample vectors arising from bandlimited, or nearly bandlimited, signals. Due to the well-known problem of spectral leakage, well-concentrated spectra do not remain well-concentrated when a finite set of samples are analyzed with the DFT. Though techniques such

as windowing can be used to mitigate spectral leakage to some degree, an alternative is to analyze signals instead using a basis of timelimited discrete prolate spheroidal sequences (DPSS's). The DPSS's are a discrete version of the prolate spheroidal wave functions studied by Landau, Pollak, and Slepian in the 1960's and 1970's [1–5]. In particular, when limited in time they provide an orthonormal basis that compactly captures the energy in sampled, bandlimited signals; we refer to this basis as the Slepian basis and expound on its properties in Section 2.

In this paper, we show that there exist fast constructions for computing approximate projections onto the leading Slepian basis elements. For a signal vector of length N , the computational complexity of these algorithms is $O(N \log N \log \frac{1}{\epsilon})$, where ϵ represents the accuracy of the approximation with respect to the ideal Slepian basis. This compares favorably with the complexity of the FFT, which is $O(N \log N)$. In contrast, the complexity of straightforward projections onto the exact Slepian basis would scale with $O(WN^2)$, where $0 < W < \frac{1}{2}$ represents the digital bandwidth of the signal of interest. Our algorithms are based on low-rank approximations to the so-called *prolate matrix*, which corresponds to an operator that alternatively time- and bandlimits a signal. We define this matrix and discuss the low-rank approximations in Section 3. We also present a new non-asymptotic bound on the width of the transition region where the eigenvalues of the prolate matrix transition from 1 to 0. Section 4 then presents the resulting fast algorithms for (i) projecting a signal onto the span of the relevant Slepian basis functions, (ii) compressing a signal to the corresponding low dimension, and (iii) solving systems of linear equations involving the prolate matrix. In Section 5 below, we show how this construction can be used to solve least-squares problems that naturally arise in array processing and other signal processing applications. In Section 6, we perform an experiment to demonstrate that this construction is faster than the exact projection.

2. BACKGROUND ON THE SLEPIAN BASIS

To begin, we provide a formal definition of the Slepian basis and briefly describe some of the key results from Slepian's

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1978 paper on DPSS's [5]. Given any $N \in \mathbb{N}$ and $W \in (0, \frac{1}{2})$, the DPSS's are a collection of N discrete-time sequences that are strictly bandlimited to the digital frequency range $|f| \leq W$ yet highly concentrated in time to the index range $n = 0, 1, \dots, N - 1$. The DPSS's are defined to be the eigenvectors of a two-step procedure in which one first time-limits the sequence and then bandlimits the sequence. Before we can state a more formal definition, let us note that for a given discrete-time signal $x[n]$, we let

$$X(f) = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi fn}$$

denote the *discrete-time Fourier transform* (DTFT) of $x[n]$. Next, we let \mathcal{B}_W denote an operator that takes a discrete-time signal, bandlimits its DTFT to the frequency range $|f| \leq W$, and returns the corresponding signal in the time domain. Additionally, we let \mathcal{T}_N denote an operator that takes an infinite-length discrete-time signal and zeros out all entries outside the index range $\{0, 1, \dots, N - 1\}$ (but still returns an infinite-length signal). With these definitions, the DPSS's are defined in [5] as follows.

Definition 1. *Given any $N \in \mathbb{N}$ and $W \in (0, \frac{1}{2})$, the DPSS's are a collection of N real-valued discrete-time sequences $s_{N,W}^{(0)}, s_{N,W}^{(1)}, \dots, s_{N,W}^{(N-1)}$ that, along with the corresponding scalar eigenvalues $1 > \lambda_{N,W}^{(0)} > \lambda_{N,W}^{(1)} > \dots > \lambda_{N,W}^{(N-1)} > 0$, satisfy*

$$\mathcal{B}_W(\mathcal{T}_N(s_{N,W}^{(\ell)})) = \lambda_{N,W}^{(\ell)} s_{N,W}^{(\ell)} \quad (1)$$

for all $\ell \in \{0, 1, \dots, N - 1\}$. The DPSS's are normalized so that

$$\|\mathcal{T}_N(s_{N,W}^{(\ell)})\|_2 = 1 \quad (2)$$

for all $\ell \in \{0, 1, \dots, N - 1\}$.

One of the central contributions of [5] was to examine the behavior of the eigenvalues $\lambda_{N,W}^{(0)}, \lambda_{N,W}^{(1)}, \dots, \lambda_{N,W}^{(N-1)}$. In particular, [5] shows that the first $2NW$ eigenvalues tend to cluster extremely close to 1, while the remaining eigenvalues tend to cluster similarly close to 0. This is made more precise in the following lemma from [5].

Lemma 1. *Suppose that W is fixed, and let $\rho \in (0, 1)$ be fixed. Then there exist constants C_0 and N_0 (which may depend on W and ρ) such that*

$$\lambda_{N,W}^{(\ell)} \geq 1 - e^{-C_0 N} \quad (3)$$

for all $\ell \leq 2NW(1 - \rho)$ and all $N \geq N_0$. Similarly, for any fixed $\rho \in (0, \frac{1}{2W} - 1)$ there exist constants C_1 and N_1 (which may depend on W and ρ) such that

$$\lambda_{N,W}^{(\ell)} \leq e^{-C_1 N} \quad (4)$$

for all $\ell \geq 2NW(1 + \rho)$ and all $N \geq N_1$.

This tells us that the range of the operator $\mathcal{B}_W \mathcal{T}_N$ has an effective dimension of $\approx 2NW$. Moreover, with only a few exceptions near the "transition region" at $\ell \approx 2NW$, we can reasonably approximate the eigenvalues $\lambda_{N,W}^{(\ell)}$ to be either 1 or 0. This will play a central role throughout our analysis.

Finally, we also note that while each DPSS actually has infinite support in time, several very useful properties hold for the collection of signals one obtains by time-limiting the DPSS's to the index range $n = 0, 1, \dots, N - 1$. First, it can be shown that [5]

$$\|\mathcal{B}_W(\mathcal{T}_N(s_{N,W}^{(\ell)}))\|_2 = \sqrt{\lambda_{N,W}^{(\ell)}}. \quad (5)$$

Comparing (2) with (5), we see that for values of ℓ where $\lambda_{N,W}^{(\ell)} \approx 1$, nearly all of the energy in $\mathcal{T}_N(s_{N,W}^{(\ell)})$ is contained in the frequencies $|f| \leq W$. While by construction the DTFT of any DPSS is perfectly bandlimited, the DTFT of the corresponding time-limited DPSS will only be concentrated in the bandwidth of interest for the first $\approx 2NW$ DPSS's. As a result, we will frequently be primarily interested in roughly the first $2NW$ DPSS's. Second, the time-limited DPSS's are orthogonal [5] so that for any $\ell, \ell' \in \{0, 1, \dots, N - 1\}$ with $\ell \neq \ell'$,

$$\langle \mathcal{T}_N(s_{N,W}^{(\ell)}), \mathcal{T}_N(s_{N,W}^{(\ell')}) \rangle = 0. \quad (6)$$

Finally, like the DPSS's, the time-limited DPSS's have a special eigenvalue relationship with the time-limiting and bandlimiting operators. In particular, if we apply the operator \mathcal{T}_N to both sides of (1), we see that the sequences $\mathcal{T}_N(s_{N,W}^{(\ell)})$ are actually eigenfunctions of the two-step procedure in which one first bandlimits a sequence and then time-limits the sequence.

These properties, together with the fact that our focus is primarily on providing computational tools for finite-length vectors, motivate our definition of the Slepian basis to be the restriction of the (time-limited) DPSS's to the index range $n = 0, 1, \dots, N - 1$ (discarding the zeros outside this range).

Definition 2. *Given any $N \in \mathbb{N}$ and $W \in (0, \frac{1}{2})$, the Slepian basis is given by the vectors $\mathbf{s}_{N,W}^{(0)}, \mathbf{s}_{N,W}^{(1)}, \dots, \mathbf{s}_{N,W}^{(N-1)} \in \mathbb{R}^N$ which are defined by restricting the time-limited DPSS's to the index range $n = 0, 1, \dots, N - 1$:*

$$\mathbf{s}_{N,W}^{(\ell)}[n] := \mathcal{T}_N(s_{N,W}^{(\ell)})[n] = s_{N,W}^{(\ell)}[n]$$

for all $\ell, n \in \{0, 1, \dots, N - 1\}$. For simplicity, we will often use the notation $\mathbf{S}_{N,W}$ to denote the $N \times N$ matrix given by

$$\mathbf{S}_{N,W} = \begin{bmatrix} \mathbf{s}_{N,W}^{(0)} & \dots & \mathbf{s}_{N,W}^{(N-1)} \end{bmatrix}.$$

Observe that combining (2) and (6), it follows that $\mathbf{S}_{N,W}$ does indeed form an orthonormal basis for \mathbb{C}^N (or for \mathbb{R}^N). However, following from our discussion above, the partial Slepian basis constructed using just the first $\approx 2NW$ basis elements will play a special role and can be shown to be remarkably effective for capturing the energy in a length- N window

of samples of a bandlimited signal (see [6] for further discussion). In such situations, we will also use the notation \mathbf{S}_K to denote the first K columns of $\mathbf{S}_{N,W}$ (where N and W are clear from the context and typically $K \approx 2NW$).

3. ANALYSIS AND APPROXIMATIONS OF THE SLEPIAN BASIS

In our discussion above we derived the Slepian basis by following the same approach as in [5] and considering the time-limitations of the eigenfunctions of the operator given by $\mathcal{B}_W \mathcal{T}_N$. It is easy to show that an alternative way to derive $\mathbf{S}_{N,W}$ is to consider the eigenvectors of the $N \times N$ prolate matrix $\mathbf{B}_{N,W}$ [7], which is the matrix with entries given by

$$\mathbf{B}_{N,W}[m, n] := \frac{\sin 2\pi W(m-n)}{\pi(m-n)} \quad (7)$$

for all $m, n \in \{0, 1, \dots, N-1\}$. Indeed, $\mathbf{B}_{N,W}$ can be understood as the finite truncation of the infinite matrix representation of $\mathcal{B}_W \mathcal{T}_N$. Thus, $\mathbf{S}_{N,W}$ contains the eigenvectors of $\mathbf{B}_{N,W}$ and we can write $\mathbf{B}_{N,W}$ as

$$\mathbf{B}_{N,W} = \mathbf{S}_{N,W} \mathbf{\Lambda}_{N,W} \mathbf{S}_{N,W}^*$$

where $\mathbf{\Lambda}_{N,W}$ is an $N \times N$ diagonal matrix with the eigenvalues $\lambda_{N,W}^{(0)}, \dots, \lambda_{N,W}^{(N-1)}$, along the main diagonal (sorted in descending order).

Our primary goal is to develop fast algorithms for working with $\mathbf{S}_{N,W}$ (or $\mathbf{B}_{N,W}$, which also arises in many practical applications). Towards this end, we will begin by examining the relationship between $\mathbf{B}_{N,W}$ and the matrix obtained by projecting onto the lowest $\approx 2NW$ Fourier coefficients. To be more precise, for any $f \in [-\frac{1}{2}, \frac{1}{2}]$ we will let

$$\mathbf{e}_f := \begin{bmatrix} e^{j2\pi f 0} \\ e^{j2\pi f 1} \\ \vdots \\ e^{j2\pi f (N-1)} \end{bmatrix}$$

denote a length- N vector of samples from a discrete-time complex exponential signal with digital frequency f . We then define W' such that $2NW'$ is the nearest odd integer to $2NW$, and we let $\mathbf{F}_{N,W}$ denote the partial Fourier matrix with the lowest $2NW'$ frequency DFT vectors of length N , i.e.,

$$\mathbf{F}_{N,W} = [\mathbf{e}_{-(2NW'-1)/2N} \quad \cdots \quad \mathbf{e}_{(2NW'-1)/2N}].$$

Note that the projection onto the span of $\mathbf{F}_{N,W}$ is given by the matrix $\mathbf{F}_{N,W} \mathbf{F}_{N,W}^*$, which has entries given by

$$\begin{aligned} [\mathbf{F}_{N,W} \mathbf{F}_{N,W}^*][m, n] &= \sum_{k=-NW'+\frac{1}{2}}^{NW'-\frac{1}{2}} e^{j2\pi(m-n)k/N} \\ &= \frac{\sin(2\pi W'(m-n))}{N \sin(\pi \frac{m-n}{N})} \end{aligned} \quad (8)$$

for $m, n \in \{0, 1, \dots, N-1\}$. Comparing (7) with (8) we see that $\mathbf{B}_{N,W}$ and $\mathbf{F}_{N,W} \mathbf{F}_{N,W}^*$ share a somewhat similar structure, where $\mathbf{B}_{N,W}$ is a Toeplitz matrix with rows (or columns) given by the sinc function, whereas $\mathbf{F}_{N,W} \mathbf{F}_{N,W}^*$ is a circulant matrix with rows (or columns) given by the digital sinc or Dirichlet function. In Theorem 1, we show that up to a small approximation error ϵ , the difference between these two matrices has a rank of $O(\log N \log \frac{1}{\epsilon})$.

Theorem 1. *Let $N \in \mathbb{N}$ and $W \in (0, \frac{1}{2})$ be given. Then for any $\epsilon \in (0, \frac{1}{2})$, there exist $N \times R_L$ matrices $\mathbf{L}_1, \mathbf{L}_2$ and an $N \times N$ matrix \mathbf{E}_F such that*

$$\mathbf{B}_{N,W} = \mathbf{F}_{N,W} \mathbf{F}_{N,W}^* + \mathbf{L}_1 \mathbf{L}_2^* + \mathbf{E}_F,$$

where

$$R_L \leq \left(\frac{4}{\pi^2} \log(8N) + 6 \right) \log \left(\frac{15}{\epsilon} \right) \quad \text{and} \quad \|\mathbf{E}_F\| \leq \epsilon.$$

Due to space limitations, the proof of this and the remaining results will be deferred to an upcoming publication. We also note that the proof of Theorem 1 provides an explicit construction of the matrices \mathbf{L}_1 and \mathbf{L}_2 , which could be of use in practice.

An important consequence of Theorem 1 which will be useful to us, and which is also of independent interest, is that it can be used to establish a nonasymptotic bound on the number of eigenvalues $\lambda_{N,W}^{(\ell)}$ of $\mathbf{B}_{N,W}$ in the ‘‘transition region’’ between ϵ and $1-\epsilon$. In particular, Lemma 1 tells us that in the limit as $N \rightarrow \infty$ we will have that the first $\approx 2NW$ eigenvalues will approach 1 while the last $\approx N(1-2W)$ eigenvalues will approach 0. However, this does not address precisely how many eigenvalues we can expect to find between ϵ and $1-\epsilon$. One can show that for any fixed $\epsilon \in (0, \frac{1}{2})$ and $W \in (0, \frac{1}{2})$, the number of eigenvalues such that $\epsilon \leq \lambda_{N,W}^{(\ell)} \leq 1-\epsilon$ is $O(\log N \log \frac{1}{\epsilon})$ as $N \rightarrow \infty$. In [5], it is shown that for any constant b , if we fix $k = \lfloor 2NW + \frac{b}{\pi} \log N \rfloor$ and let $N \rightarrow \infty$, then $\lambda_{N,W}^{(k)} \rightarrow (1 + e^{\pi b})^{-1}$. By setting $b = \frac{1}{\pi} \log(\frac{1}{\epsilon} - 1)$, we get $\lambda_{N,W}^{(k)} \rightarrow \epsilon$. Similarly, setting $b = -\frac{1}{\pi} \log(\frac{1}{\epsilon} - 1)$ yields $\lambda_{N,W}^{(k)} \rightarrow 1-\epsilon$. This gives the following asymptotic result:

$$\#\{\ell : \epsilon \leq \lambda_{N,W}^{(\ell)} \leq 1-\epsilon\} \sim \frac{2}{\pi^2} \log N \log \left(\frac{1}{\epsilon} - 1 \right).$$

A nonasymptotic bound on the width of this transition region is given in [8], which shows that for any $N \in \mathbb{N}$, $W \in (0, \frac{1}{2})$, and $\epsilon \in (0, \frac{1}{2})$,

$$\#\{\ell : \epsilon \leq \lambda_{N,W}^{(\ell)} \leq 1-\epsilon\} \leq \frac{\frac{2}{\pi^2} \log(N-1) + \frac{2}{\pi^2} \frac{2N-1}{N-1}}{\epsilon(1-\epsilon)}.$$

This bound correctly highlights the logarithmic dependence on N , but can be quite poor when ϵ is very small ($O(1/\epsilon)$) as opposed to the $O(\log(1/\epsilon))$ dependence in the asymptotic

result). In the following corollary of Theorem 1, we significantly sharpen this bound in terms of its dependence on ϵ to within a constant factor of the optimal asymptotic result. The intuition behind this result is that Theorem 1 demonstrates that $\mathbf{B}_{N,W}$ can be approximated as $\mathbf{F}_{N,W}\mathbf{F}_{N,W}^*$ (a matrix whose eigenvalues are all either equal to 1 or 0) plus a low-rank correction, and the rank of this correction limits the number of possible eigenvalues in the transition region.

Corollary 1. For any $N \in \mathbb{N}$, $W \in (0, \frac{1}{2})$, and $\epsilon \in (0, \frac{1}{2})$,

$$\#\{\ell : \epsilon < \lambda_{N,W}^{(\ell)} < 1 - \epsilon\} \leq \left(\frac{8}{\pi^2} \log(8N) + 12 \right) \log \left(\frac{15}{\epsilon} \right).$$

Finally, we describe one additional consequence of these results. Recall that $\mathbf{B}_{N,W} = \mathbf{S}_{N,W}\mathbf{\Lambda}_{N,W}\mathbf{S}_{N,W}^*$. From Corollary 1 we have that the diagonal entries of the matrix $\mathbf{\Lambda}_{N,W}$ are mostly very close to 1 or 0, with only a small number of eigenvalues lying in between. Thus, recalling that \mathbf{S}_K denotes the $N \times K$ matrix containing the first K elements of the Slepian basis $\mathbf{S}_{N,W}$, it is reasonable to expect that $\mathbf{B}_{N,W}$ and $\mathbf{S}_K\mathbf{S}_K^*$ (the matrix obtained by setting the top K eigenvalues to 1 and the remainder to 0) should be within a low-rank correction when $K \approx 2NW$. The following corollary shows that this is indeed the case.

Corollary 2. Let $N \in \mathbb{N}$ and $W \in (0, \frac{1}{2})$ be given. For any $\epsilon \in (0, \frac{1}{2})$, fix K to be such that $\lambda_{N,W}^{(K-1)} > \epsilon$ and $\lambda_{N,W}^{(K)} < 1 - \epsilon$. Then there exist $N \times R_U$ matrices $\mathbf{U}_1, \mathbf{U}_2$ and an $N \times N$ matrix \mathbf{E}_S such that

$$\mathbf{S}_K\mathbf{S}_K^* = \mathbf{B}_{N,W} + \mathbf{U}_1\mathbf{U}_2^* + \mathbf{E}_S,$$

where

$$R_U \leq \left(\frac{8}{\pi^2} \log(8N) + 12 \right) \log \left(\frac{15}{\epsilon} \right) \text{ and } \|\mathbf{E}_S\| \leq \epsilon.$$

4. FAST COMPUTATIONS WITH THE APPROXIMATE SLEPIAN BASIS

Suppose we wish to compress a vector $\mathbf{x} \in \mathbb{C}^N$ of N uniformly spaced samples of a signal down to a vector of $K \approx 2NW$ elements in such a way that best preserves the DTFT of the signal over $|f| \leq W$. We can do this by storing $\mathbf{S}_K^*\mathbf{x}$, which is a vector of $K < N$ elements, and then later recovering $\mathbf{S}_K\mathbf{S}_K^*\mathbf{x}$, which contains nearly all of the energy of the signal in the frequency band $|f| \leq W$. However, naïve multiplication of \mathbf{S}_K or \mathbf{S}_K^* takes $O(NK) = O(2WN^2)$ operations. For certain applications, this may be intractable.

The following theorem gives us a way to approximate this method of data compression/recovery using $O(N \log N \log \frac{1}{\epsilon})$ operations.

Theorem 2. Let $N \in \mathbb{N}$ and $W \in (0, \frac{1}{2})$ be given. Fix an integer K and a tolerance $\epsilon \in (0, \frac{1}{2})$ such that $\lambda_{N,W}^{(K-1)} > \epsilon$

and $\lambda_{N,W}^{(K)} < 1 - \epsilon$. Then there exist $N \times K'$ matrices $\mathbf{T}_1, \mathbf{T}_2$, where

$$K' \leq \lceil 2NW \rceil + \left(\frac{12}{\pi^2} \log(8N) + 18 \right) \log \left(\frac{15}{\epsilon} \right),$$

such that $\|\mathbf{T}_1\mathbf{T}_2^* - \mathbf{S}_K\mathbf{S}_K^*\| \leq 2\epsilon$, and \mathbf{T}_1 and \mathbf{T}_2^* can be applied to a vector in $O(N \log N \log \frac{1}{\epsilon})$ operations.

The proof of Theorem 2 follows by combining Theorem 1 and Corollary 2. Specifically, we can set $\mathbf{T}_1 = [\mathbf{F}_{N,W} \ \mathbf{L}_1 \ \mathbf{U}_1]$ and $\mathbf{T}_2 = [\mathbf{F}_{N,W} \ \mathbf{L}_2 \ \mathbf{U}_2]$. The matrices $\mathbf{F}_{N,W}$ and $\mathbf{F}_{N,W}^*$ can be applied in $O(N \log N)$ operations via the FFT. Also, the matrices $\mathbf{L}_1, \mathbf{L}_2, \mathbf{U}_1, \mathbf{U}_2$ are of size $N \times O(\log N \log \frac{1}{\epsilon})$, and so, $\mathbf{L}_1, \mathbf{L}_2, \mathbf{U}_1, \mathbf{U}_2^*$ can be applied in $O(N \log N \log \frac{1}{\epsilon})$ operations.

Alternatively, if we only require computing the projected vector $\mathbf{S}_K\mathbf{S}_K^*\mathbf{x}$, and compression is not required, then there is a simpler solution, given by the following theorem.

Theorem 3. Let $N \in \mathbb{N}$ and $W \in (0, \frac{1}{2})$ be given. Fix an integer K and a tolerance $\epsilon \in (0, \frac{1}{2})$ such that $\lambda_{N,W}^{(K-1)} > \epsilon$ and $\lambda_{N,W}^{(K)} < 1 - \epsilon$. Then, there exists an $N \times N$ matrix $\widehat{\mathbf{P}}_K$ such that $\|\widehat{\mathbf{P}}_K - \mathbf{S}_K\mathbf{S}_K^*\| \leq \epsilon$, and $\widehat{\mathbf{P}}_K$ can be applied to a vector in $O(N \log N \log \frac{1}{\epsilon})$ operations.

The proof of Theorem 3 follows almost immediately from Corollary 2. Specifically, we let $\widehat{\mathbf{P}}_K = \mathbf{B}_{N,W} + \mathbf{U}_1\mathbf{U}_2^*$. This matrix can be applied to a vector in $O(N \log N \log \frac{1}{\epsilon})$ operations because it is the sum of a Toeplitz matrix and a factored low rank matrix. The low rank update simply adjusts the eigenvalues of $\mathbf{B}_{N,W}$ which are not within ϵ of 1 or 0.

A closely related problem to working with the matrix $\mathbf{S}_K\mathbf{S}_K^*$ concerns the task of solving a linear system of the form $\mathbf{y} = \mathbf{B}_{N,W}\mathbf{x}$. Since the prolate matrix has several eigenvalues that are close to 0, the system is often solved by using the rank- K truncated pseudoinverse of $\mathbf{B}_{N,W}$ where $K \approx 2NW$. Even if the pseudoinverse is precomputed and factored ahead of time, it still takes $O(NK) = O(2WN^2)$ operations to apply to a vector \mathbf{y} . The following theorem gives us a way to approximately apply the truncated pseudoinverse of $\mathbf{B}_{N,W}$ using $O(N \log N \log \frac{1}{\epsilon})$ operations.

Theorem 4. Let $N \in \mathbb{N}$ and $W \in (0, \frac{1}{2})$ be given. Fix an integer K and a tolerance $\epsilon \in (0, \frac{1}{2})$ such that $\lambda_{N,W}^{(K-1)} > \epsilon$ and $\lambda_{N,W}^{(K)} < 1 - \epsilon$. Let $\mathbf{B}_{N,W}^\dagger$ be the rank- K truncated pseudoinverse of $\mathbf{B}_{N,W}$. Then, there exists an $N \times N$ matrix $\widehat{\mathbf{B}}_{N,W}^\dagger$ such that $\|\widehat{\mathbf{B}}_{N,W}^\dagger - \mathbf{B}_{N,W}^\dagger\| \leq 3\epsilon$, and $\widehat{\mathbf{B}}_{N,W}^\dagger$ can be applied to a vector in $O(N \log N \log \frac{1}{\epsilon})$ operations.

The proof of this theorem is similar to the proof of Theorem 3. Specifically, we let $\widehat{\mathbf{B}}_{N,W}^\dagger = \mathbf{B}_{N,W} + \mathbf{V}_1\mathbf{V}_2^*$, where $\mathbf{V}_1\mathbf{V}_2^*$ is a low rank correction which adjusts the eigenvalues of $\mathbf{B}_{N,W}$ which are not within ϵ of 1 or 0. Note that $\mathbf{V}_1, \mathbf{V}_2$ are the same size as $\mathbf{U}_1, \mathbf{U}_2$.

5. SOLVING LEAST-SQUARES PROBLEMS

The fast projection techniques discussed above give us an efficient way to manipulate a set of consecutive samples of a bandlimited function. We first consider the prototype problem of estimating a continuous-time signal $x_c(t)$ from a finite number of equally spaced samples. If $x_c(t)$ is bandlimited to F Hz, it can be represented without loss from the samples $x_d[n] = x_c(nT)$ for all $n \in \mathbb{Z}$, when $TF =: W < 1/2$. Let $\mathcal{I}_N : \ell_2(\mathbb{Z}) \rightarrow \mathbb{C}^N$ denote the index-limiting operator that restricts a sequence to its entries on $n = 0, 1, \dots, N-1$ (and produces a vector of length N). If we observe N of these samples, at $n = 0, \dots, N-1$, then our observations can be written as

$$\mathbf{y} = \mathcal{I}_N(x_d) = \mathcal{I}_N(\mathcal{B}_W(x_d)) = \mathcal{A}(x_d),$$

where we have combined the bandlimiting and index-limiting operators into one linear operator $\mathcal{A} : \ell_2(\mathbb{Z}) \rightarrow \mathbb{C}^N$. Given \mathbf{y} , we would like the least-squares estimate of x_d :

$$\underset{x \in \ell_2(\mathbb{Z})}{\text{minimize}} \sum_{n=-\infty}^{\infty} |x[n]|^2 \quad \text{subject to} \quad \mathcal{A}(x) = \mathbf{y}.$$

The solution to this problem is given by the pseudo-inverse of \mathcal{A} , $\hat{x} = \mathcal{A}^\dagger \mathbf{y}$. It is easy to check that $\hat{x} = \mathcal{A}^\dagger \mathbf{y} = \mathcal{A}^*(\mathcal{B}_{N,W}^\dagger \mathbf{y})$, where \mathcal{A}^* is the adjoint of \mathcal{A} :

$$[\mathcal{A}^*(\mathbf{v})][m] = \sum_{k=0}^{N-1} v[k] \frac{\sin(2\pi W(m-k))}{2\pi W(m-k)}.$$

So the N -vector $\mathcal{B}_{N,W}^\dagger \mathbf{y}$ completely defines the infinite-length sequence that is least-squares optimal; with $\mathcal{B}_{N,W}^\dagger \mathbf{y}$ in hand, we can use the expression above to compute as many samples of \hat{x} as we like. Furthermore, for any \mathbf{v} , the mapping from \mathbf{v} to M samples of $\mathcal{A}^*(\mathbf{v})$ itself specified by a partial Toeplitz matrix, and can be implemented efficiently. Theorem 4 above shows us that the key computation of $\mathcal{B}_{N,W}^\dagger \mathbf{y}$ can be computed (to a very good approximation) in $O(N \log N \log \frac{1}{\epsilon})$ time.

A key application of the least-squares problem comes from imaging a reflectivity field $z(\theta)$, $\theta \in [-\pi/2, \pi/2]$ using a linear array with N elements that are equally spaced. If the scene is probed by emitting a narrow-band sinusoid from a fixed point (the array origin, say), the response at array element n , for $n = 0, \dots, N-1$ can be interpreted as a sample of the continuous-time Fourier transform of $z'(\tau) = z(\arcsin(2\tau))/\sqrt{1-4\tau^2}$, $\tau \in [-1/2, 1/2]$. Since this function is supported only on $[-1/2, 1/2]$, its Fourier transform is “bandlimited”, and can be estimated in its entirety by solving the least-squares problem above.

6. NUMERICS

To test our fast projection method, we fix the half-bandwidth $W = \frac{1}{4}$ and the tolerance $\epsilon = 10^{-3}$, and vary the signal length N over several values between 64 and 4096.

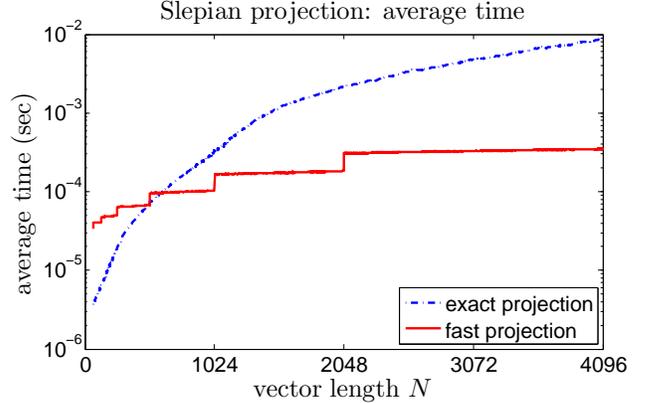


Fig. 1. Average time needed for the naïve exact and fast approximate Slepian projections.

For each value of N we randomly generate several length- N vectors and project each one onto the span of the first $K = \text{round}(2NW)$ elements of the Slepian basis using both the exact projection matrix $\mathcal{S}_K \mathcal{S}_K^*$ and the fast projection matrix $\hat{\mathcal{P}}_K = \mathcal{B}_{N,W} + \mathcal{U}_1 \mathcal{U}_2^*$. The prolate matrix is applied to \mathbf{x} via an FFT whose length is the smallest power of 2 that is at least $2N$. Over all values of N , and all randomly generated vectors \mathbf{x} , the maximum relative error $\|\hat{\mathcal{P}}_K \mathbf{x} - \mathcal{S}_K \mathcal{S}_K^* \mathbf{x}\|_2 / \|\mathbf{x}\|_2$ was less than 4.2×10^{-4} . A plot of the average time needed to project a vector onto the span of \mathcal{S}_K using both the exact projection and the fast projection is shown in Figure 1. The time required by the exact projection grows quadratically with N , while the time required by the fast projection grows a bit faster than linearly in N .

7. REFERENCES

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