550.643 Project 1: Marginals probabilities using Belief Propagation.

The solution must be written like you would write a report, with explanations and commented results. A series of numbers and figures is not sufficient.

There are four problems.

Only the program sources for Problem 1 are needed.

Due on Wednesday March 26.

Problem 1

(1) Implement a program running belief propagation for an acyclic graph with N nodes, and for a process $X = (X_s, s = 1, \ldots, N)$ a process with q states (nodes are labelled from 1 to N). The underlying probability is

$$\pi(x) = P(X = x) = \frac{1}{Z} \prod_{s \in V} \varphi_x(x_s) \prod_{\{s,t\} \in E} \varphi_{s,t}(x_s, x_t).$$

The program will have the following input and output:

- Input:
  - An M by 2 list, E, of edges (M is the number of edges). Each row contain two node indices corresponding to the edges. Because messages go both ways, it will be convenient to make the array redundant and contain both (s, t) and (t, s).
  - An N by q array providing univariate functions $(\varphi_s(a), s = 1, \ldots, N, a = 1, \ldots, q)$.
  - An M by q by q array providing bivariate functions $(\varphi_{e_k}(a, b), k = 1, \ldots, M, a, b = 1, \ldots, q)$, where $e_k = (s_k, t_k)$ is the kth edge in the first array.
- Output:
  - An M by q list containing the messages, $(m_{e_k}(a), k = 1, \ldots, M, a = 1, \ldots, q$.
  - The associated list of probabilities $(\pi_s(a), s = 1, \ldots, N, a = 1, \ldots, q)$ estimated according to the belief propagation formulae.

Design the implementation so $\sum_{a=1}^{q} = m_{st}(a) = 1$ for all messages at all steps, and let messages be updated iteratively in the order the edges are listed.

(2) For a fixed k, we can consider $\varphi_{e_k}(., .)$ as a q by q matrix Φ_k. Give the relation between Φ_k and Φ_l, when $e_k = (s, t)$ amd $e_l = (t, s)$.

Problem 2

In this problem, G is the acyclic graph with $N = 2^d$ vertices, and edges given by

$$E = \{\{k, [(k+1)/2]\}, \{k, 2k-1\}, \{k, 2k\}, k = 2, \ldots, N/2\}$$

where $[u]$ is the integer part of u.

The state space is $F = \{1, 2\}$.

The interactions are such that matrices Φ_k above are all equal and symmetric with

$$\Phi_k = \Phi = \begin{pmatrix} 1+\theta & 1-\theta \\ 1-\theta & 1+\theta \end{pmatrix}$$

with $\theta \in (-1, +\infty)$.

The univariate functions are $\varphi_s(a) = 1$ for all s and a.

(1) Draw the graph G for $d = 2$ and 3. What are the leaves of G (vertices with only one neighbor).

(2) Prove that the model coincides (up to a relabelling of the state space) with an Ising model on G.

(3) Compute (with an analytical proof), the marginal probability of X_1, $\pi_1(1)$.

(4) We take $\theta = 1/N$ and assume that the values of X_s are observed for all s with labels $N/2+1, \ldots N$. Use the program written in Problem 1 to estimate the conditional probability

$$P(X_1 = 1 | X_s = x_s, s \geq N/2 + 1)$$

when $d = 5, 10, 15$ in the following cases:

- $X_s = 1$ if k is a multiple of 3, $X_s = 2$ otherwise, with $s = N/2 + k$, $k = 1, \ldots, N/2$.
- $X_s = 1$ if $k \leq 2N/6$, $X_s = 2$ otherwise, with $s = N/2 + k$, $k = 1, \ldots, N/2$.

Describe the input given to the algorithm.

Problem 3.

We consider the single-loop graph G with N vertices and edges $\{k, k+1\}$ for $k = 1, \ldots, N-1$, and $\{N, 1\}$.

We assume $\varphi_{k,k+1}(a, b) = 1$ if $a = b$ and $\varphi_{k,k+1}(a, b) = 1.5 + \sin(k\pi/N)$ for $k = 1, \ldots, N$ (with $N + 1 \equiv 1$). Also, $\varphi_k(1) = 1$ and $\varphi_k(2) = 2$ for all k.

(1) Run the belief propagation algorithm for this process with $N = 10, 20, 100$. Describe the input given to the algorithm of Problem 1. Check experimentally that the messages converge. Plot the estimated marginal probabilities of state 1, $\pi_k(1)$ in function of the vertex index, k.

(2) We want to compute exact probabilities. Show that, for $k > 1$, π_k and π_1 can be computed analytically in function of the conditional probabilities $P(X_k = a|X_1 = b)$ and $P(X_1 = a|X_k = b)$ for $a, b \in F$.

(Hint: try to express the ratio $P(X_1 = a, X_k = b)/P(X_1 = 1, X_k = 1)$ in function of the conditional probabilities.)

(3) Explain why the belief propagation algorithm reliably computes the conditional probabilities in (2). Use this to provide exact expressions for π_k, for $k = 1, 2, N/2$ and $N = 10, 20, 100$. Describe the input given to the algorithm of Problem 1.

Compare the results to those obtained in (1).

Problem 4.

We now consider the loopy graph G with $N = 4n - 1$ vertices and edges $\{k, k+1\}$ for $k = 1, \ldots, 3n-1$ and $k = 3n+1, \ldots, 4n-2$, plus edges $\{3n, n\}, \{n, 3n+1\}, \{4n-1, 1\}$.

We assume $\varphi_{k,k+1}(a,b) = 1$ if $a = b$ and $\varphi_{k,k+1}(a,b) = .75 + \sin(k\pi/N)/2$ for $k = 1, \ldots, N-1, k \neq 3n$ (with $N + 1 \equiv 1$). We let

$$\varphi_{3n,n}(a,b) = \varphi_{n,3n+1}(a,b) = 1.25 \text{ if } a \neq b$$

and 1 if $a = b$.

Finally, $\varphi_k(1) = 1$ and $\varphi_k(2) = 2$ for all k.

Use the ideas developped in Problem 3-3 (conditioning on two vertices instead of one) to compute exactly the probabilities π_1, π_n and π_{2n-1} for $n = 25$. Compare them to those obtained with belief propagation. (Describe the input given to the algorithm of Problem 1 each time you used it.)