

THE METRIC SPACES, EULER EQUATIONS, AND NORMAL GEODESIC IMAGE MOTIONS OF COMPUTATIONAL ANATOMY

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1. INTRODUCTION

Over the past several years our group has been studying Biological shape in the emerging new discipline of Computational Anatomy (CA) [1]. CA consists of several components: (i) the construction of coordinatized anatomical manifolds, (ii) comparison of anatomical manifolds, and (iii) inference of morphometric change on anatomical manifolds. In this paper we focus on (ii) the comparison of anatomical shapes and structures in imagery via metric mapping. Our group has been studying natural shapes in imagery using the diffeomorphisms [2, 3, 4, 1, 5, 6] the natural extension of the finite dimensional matrix groups. Shapes and imagery are studied as an orbit via the action of diffeomorphic transformation.

Define the diffeomorphisms $g \in \mathcal{G}$ on the background space $X \subset \mathbb{R}^d$ generated via the flows $g_0 = id, g = g_1, g_t$ satisfying

$$\frac{dg_t}{dt} = v_t \circ g_t \quad t \in [0, 1]. \quad (1)$$

The anatomical orbit or deformable template is constructed via the action of the group according to

$$\mathcal{J} = \mathcal{G} \cdot I = \{I = I_\alpha \circ g, g \in \mathcal{G}\}; \quad (2)$$

oftentime I_α 's are called the exemplars or templates, with the orbit the deformable template.

The anatomical orbit or deformable template is made into a metric space with a metric distance $\rho : \mathcal{J} \times \mathcal{J} \rightarrow \mathbb{R}^+$ by solving for the geodesic

flow connecting two images in the orbit, and measuring it's length or square-root energy:

$$\rho^2(I_0, I_1) = \inf_{v: g_t = v_t(g_t)} \int_0^1 \|v_t\|_L^2 dt \text{ s.t. } I_0 \circ g_1^{-1} = I_1, \quad (3)$$

where $\|\cdot\|_L$ dominates an appropriate Sobolev norm to ensure that the solutions are sufficiently smooth to generate a diffeomorphism [4, 3].

This is a principal focus in Computational Anatomy, the study of shape via high dimensional transformations generated via the flows of Eqn. 1. The notion of studying large deformations via the ODE of equation 1 was originally put forward by Christensen and Rabbitt [2]; introducing the metric structure and the associated variational problem of minimizing over all flows connecting images was first studied in [4, 3]. The metric is defined according to $\|\cdot\|_L^2 = \langle L \cdot, \cdot \rangle_2$, with L a differential operator; the finiteness of norm implies sufficient differentiability (smoothness) ensuring that the ODE integrates and is a flow of diffeomorphisms.

The purpose of this paper is to examine the generation of the geodesics associated with the metric from several points of view, the first the Euler equation describing the geodesic diffeomorphic flow, and the second the variational formulation of the geodesic in terms of the minimizing flow of vector fields which generate them. As we show, it is the property of the vector fields describing the geodesic flow which links us to the notions of geodesic normal evolution of the image. This of course links our work on geodesics

to the rapidly growing community working in *level set methods*. Interestingly, as we show, it is not the vector fields $v_t, t \in [0, 1]$ which are associated with normal geodesic motion, but rather it is $Lv_t, t \in [0, 1]$. As we shall see, the *normal geodesic motion* $Lv_t, t \in [0, 1]$ implies the velocity of motion $v_t, t \in [0, 1]$ of the shape being supported over the entire background space, thereby giving the global property that the resulting integrated vector field generates a diffeomorphism on the entire extrinsic space. This carries the smooth submanifold diffeomorphically.

1.1. Euler equation for diffeomorphisms

The Euler equation was first derived in [7] and subsequently in [6] using the calculus of variations directly. To ensure that the O.D.E.s generate diffeomorphisms Dupuis [4] and Trouvé [3] have shown that a sufficient amount of smoothness required is that the fields be at least 1-times continuously differentiable, compactly supported with $\|v\|_{1,\infty} = \sup_{x \in X} |v(x)| + \sum_{i=1}^d |\frac{\partial v}{\partial x_i}(x)|$. The construction therefore depends on the choice of the inner product $\langle \cdot, \cdot \rangle_L \doteq \langle L \cdot, \cdot \rangle_2$ so that the norm $\|\cdot\|_L < \infty$ dominates the sup norm $\|\cdot\|_{p,\infty}$ for at least $p = 1$ derivative (See [4, 2]).

Proposition 1. *Let the $\|v\|_L^2 = \langle Lv, v \rangle_2$ satisfy the $p = 1$ condition so that there exists some constant c such that for some $p \geq 1$ and all $v \in V$, $\langle Lv, v \rangle_2 \geq c\|v\|_{p,\infty}$. The geodesics of that equation 3 are solutions of the Euler equation*

$$\frac{\partial Lv}{\partial t} + (Dv)^*Lv + \text{div}(Lv \otimes v) = 0, \quad (4)$$

with D a Jacobian matrix, divergence operator $\text{div } v = \nabla^*v = \sum_{i=1}^d \frac{\partial v_i}{\partial x_i}$ and $\text{div}(Lv \otimes v) = (DLv)v + (\text{div } v)Lv$.

Proof. See [6]. □

1.2. Euler equation for Matrix Groups

To interpret the infinite dimensional Euler equation setting, examine their direct analogue of the

Euler equation for the metric length in the finite dimensional matrix Lie groups. Define the matrix valued group elements $g, h \in \mathbf{GL}(d)$, and the minimal geodesics connecting $g, h \in \mathbf{GL}(d)$ with tangent norm $\|\dot{g}_t g_t^{-1} = v_t\|_M$ minimizing $\int_0^1 \|v_t\|_M^2 dt$. For the matrix groups then $v_t \in \mathbb{R}^{d \times d}$ satisfying $\dot{g}_t = v_t g_t$, and define the norm according to

$$\|v\|_M^2 = \langle Mv, v \rangle_{\mathbb{R}^{d \times d}} = \text{trace } Mvv^*, \quad (5)$$

with $(\cdot)^*$ denoting transpose. The minimal geodesics connecting $g, h \in \mathbf{GL}(d)$ with tangent norm $\|\dot{g}_t g_t^{-1} = v_t\|_M$ minimizing $\int_0^1 \|v_t\|_M^2 dt$ satisfying $\dot{g} = vg, v, g \in \mathbb{R}^{d \times d}$ matrices with $g_0 = g, g_1 = h$ are solutions of the Euler equation:

$$\frac{dMv_t}{dt} - Mv_t v_t^* + v_t^* Mv_t = 0. \quad (6)$$

Notice the similarity between equation 4 and 6.

2. NORMAL VECTOR FIELD MOTION

The Euler equation describes the properties of the flow of the group element; now we focus on the properties of the vector fields in the Lie algebra. First we establish that it is a general property that the vector fields satisfies a normal property to the smooth level sets.

2.1. The Normal Motion property

Start with a smooth image, represented by a function I_0 defined on X , I_1 in the orbit of I_0 for the group G of diffeomorphisms. Assume the geodesic path exists connecting I_0 to I_1 ; the fact that $g = (g_t)$ is a geodesic path induces a constraint on the initial vector field v_0 . Indeed, let $g = (g_t^v)$ be such a solution. We consider a first order expansion of the solution around $t = 0$ to deduce a normality constraint on the initial momentum Lv_0 .

At first order around $t = 0$, $g_\epsilon(x) \simeq x + \epsilon v_0(x)$ so that $I_\epsilon = I_0 \circ g_\epsilon^{-1}(x) \simeq I_0 - \epsilon \langle \nabla I_0, v_0 \rangle_L$ and the cost to go from I_0 to I_ϵ is (still at first order) $\epsilon |v_0|_L$. However, any $u \in \mathfrak{g}$ such that $\langle \nabla I_0, u \rangle_{\mathbb{R}^d} = \langle \nabla I_0, v_0 \rangle_{\mathbb{R}^d}$, will lead to the same

I_ϵ so that the least deformation cost from I to I_ϵ is $\epsilon|p_{I_0}(v_0)|$ where $p_{I_0}(v_0)$ the unique solution of the minimization problem:

$$(P) \quad \begin{cases} \inf_{u \in \mathfrak{g}} \|u\|_L \text{ subject to:} \\ \langle (v_0 - u)(x), \nabla I_0(x) \rangle_{\mathbb{R}^d} = 0, \forall x \in X \end{cases} \quad (7)$$

Now, since $(g^v)_t$ is geodesic path minimizing the deformation cost from I_0 to I_1 , it minimizes also the deformation cost from I_0 to I_ϵ so that

$$v_0 \approx p_{I_0}(v_0). \quad (8)$$

Clearly p_{I_0} is an orthogonal projection. Indeed, if $\mathfrak{g}_{I_0} = \{ h \in \mathfrak{g} \mid \langle \nabla I_0(x), h(x) \rangle_{\mathbb{R}^d} = 0, \forall x \in X \}$, the constraints of \mathcal{P} can be written $u - v_0 \in V_{I_0}$ so that $p_{I_0}(v)$ is the orthogonal projection of v on $\mathfrak{g}_{I_0}^\perp$ i.e. the orthogonal space of \mathfrak{g}_{I_0} . Hence, equality (8), translates to $\forall h \in \mathfrak{g}$ such that for all $x \in X$,

$$\langle \nabla I_0(x), h(x) \rangle = 0, \text{ we have } \langle v_0, h \rangle_L = 0. \quad (9)$$

Now we state the following rigorous results for normal motion. Denote for all $t \in [0, 1]$, $p_{I_t}(v_t)$ the solution of (7) applied at time t .

Theorem 1 (Geodesic Normal Motion). *Assume that I_0 is $C^1(X, \mathbb{R})$ and that the vector fields are smooth (continuously embedded in $C^1(X, \mathbb{R}^d)$). Let $g = (g_t^v)$ be a geodesic path solution. Then, for almost all $t \in [0, 1]$*

$$v_t = p_{I_t}(v_t).$$

Proof. Denote $h_t \doteq v_t - p_{I_t}(v_t)$ and define for any $\epsilon \in [0, 1]$, $v_t^\epsilon \doteq v_t + \epsilon h_t$, and $g_t^\epsilon \doteq g_t^{v_t^\epsilon}$. We first show that for any $0 \leq t \leq 1$ $I_0 \circ g_{t,0} = I_0 \circ g_{t,0}^\epsilon$, then

$$\begin{aligned} & \frac{\partial}{\partial t} (I_0 \circ g_{t,0} \circ g_{0,t}^\epsilon)(x) \\ &= \left\langle \nabla_{g_{t,0}(g_{0,t}^\epsilon(x))} I_0, -d_{g_{0,t}^\epsilon(x)} g_{t,0}(v_t(g_{0,t}^\epsilon(x))) + \right. \\ & \quad \left. d_{g_{0,t}^\epsilon(x)} g_{t,0}((v_t + \epsilon h_t)(g_{0,t}^\epsilon(x))) \right\rangle_{\mathbb{R}^d} \\ &= \epsilon \left\langle \nabla_{g_{t,0}(g_{0,t}^\epsilon(x))} I_0, d_{g_{0,t}^\epsilon(x)} g_{t,0}(h_t(g_{0,t}^\epsilon(x))) \right\rangle_{\mathbb{R}^d} \\ &= \epsilon \left\langle \nabla_{g_{0,t}^\epsilon(x)} I_t, h_t(g_{0,t}^\epsilon(x)) \right\rangle_{\mathbb{R}^d} = 0 \end{aligned}$$

since by definition of the projection $p_{I_t}(v_t)$, we have for any $x \in X$, $\langle h_t(x), \nabla I_t(x) \rangle_{\mathbb{R}^d}$

$$= \langle v_t(x), \nabla I_t(x) \rangle_{\mathbb{R}^d} - \langle p_{I_t}(v_t), \nabla I_t(x) \rangle_{\mathbb{R}^d} = 0.$$

Then, we get $I_0 \circ g_{t,0} \circ g_{0,t}^\epsilon = I_0$ ie $I_0 \circ g_{t,0} = I_0 \circ g_{t,0}^\epsilon$. Considering $\epsilon = 1$ and $t = 1$, we deduce that $I_1 = I_0 \circ g_{1,0}^1$. However, since $\langle h_t, v_t \rangle_L = \langle v_t - p_{I_t}(v_t), v_t \rangle_L = 0$, we get $\|v_t + h_t\|_L^2 = \|v_t\|_L^2 + \|h_t\|_L^2$. Since $t \rightarrow v_t$ leads to the paths with lowest kinetic energy from I_0 to I_1 , we deduce that $\int_0^1 \|h_t\|_L^2 = 0$. \square

2.2. The Variation Problem in the Vector Fields

The general form for the vector fields associated with the geodesic motion must be normal to the level sets for smooth images. Now examine the particular form for the normal motion so that the vector fields describing the geodesics connecting particular images I_0 and I_1 in the orbit. For this we use the approach taken by Faisal Beg [8] for examining direct variation of the minimum problem in the space of vector fields (in contrast to the Euler equations describing the geodesic diffeomorphic flow).

Proposition 2 (Faisal Beg [8]). *Let the template I be continuously differentiable with $\nabla(I \circ g) = (Dg)^* \nabla I \circ g$, and data I_1 with the matching functional energy as given by*

$$\int_0^1 \|v_t\|_L^2 dt + \frac{1}{2\sigma^2} \|I_1 - I_0(g_1^{-1})\|_2^2, \quad (10)$$

with norm $\|v\|_L^2 = \langle Lv, v \rangle_2$, then the minimizer with respect to variations of the vector fields $v \rightarrow v + \epsilon \psi$ of the energy satisfies the Euler equation

$$Lv_t = -\frac{|Dg_{t,1}|}{2\sigma^2} (I_0(g_t^{-1}) - I_1(g_{t,1})) \nabla(I_0(g_t^{-1})), \quad (11)$$

with $g_{t,1} = g_1(g_t^{-1})$.

Proof. See [8, 9]. \square

3. THE NORMAL MOTION CONDITION

Equation 11 is normal property of momentum which at the origin takes the form

$$Lv_0 = -\frac{1}{\sigma^2} |Dg_{0,1}| (I_1 \circ g_{0,1} - I_0) \nabla I_0.$$

Figure 1 illustrates the geodesic motion at the origin Lv_0 generated via the Beg algorithm [8]. Panel 3 shows comparisons between the Lv_0 motion generated by the Beg algorithm and the gradient of the image ∇I_0 . Shown superimposed at every point in the grid are the arrows depicting the direction vector of each. Notice they superimpose and are concentrated near the boundary. Thus we see that Lv_0 is supported and normal to the level sets (which are concentrated at the boundary), and in the same direction as the normals. Panel 4 shows that v_0 has full support over the entire background space. Panels 3 and 4 emphasize that while the normal geodesic motion Lv_0 are highly localized, the velocity v_0 of motion extends over the entire object. This is because to guarantee the existence of diffeomorphisms, L is a power of differential operators and L^{-1} is a smoothing kernel extending over the full background space.

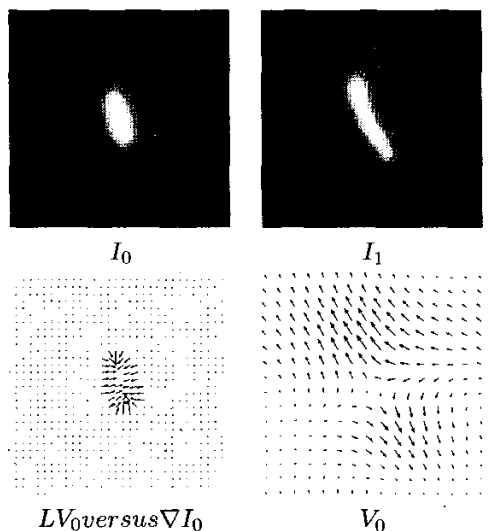


Fig. 1. Panels 1 and 2 show the two mitochondria shapes I_0, I_1 . Panel 3 shows Lv_0 versus ∇I_0 superimposed; panel 4 shows V_0 .

4. REFERENCES

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