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ON A CLASS OF DIFFEOMORPHIC MATCHING PROBLEMS IN ONE DIMENSION*

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Abstract. We study a class of functional which can be used for matching objects which can be represented as mappings from a fixed interval, I, to some "feature space." This class of functionals corresponds to "elastic matching" in which a symmetry condition and a "focus invariance" are imposed. We provide sufficient conditions under which an optimal matching can be found between two such mappings, the optimal matching being a homeomorphism of the interval I. The differentiability of this matching is also studied, and an application to plane curve comparison is provided.

Key words. calculus of variations, shape representation and recognition, elastic matching, geodesic distance

AMS subject classifications. Primary, 49J45; Secondary, 68T10, 53A04

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1. Introduction.

In many applications, objects of interest can be represented as numerical functions θ which are defined on some interval $I \subset \mathbb{R}$ and take values in $\mathbb{R}^d$. Several examples may come from signal processing, in which measurements are made during a certain time interval (e.g., speech recognition), analysis of one-dimensional (1D) geological data (e.g., measurements in wells, I being a depth interval), or shape recognition, in which an object can represent a two-dimensional (2D) or three-dimensional (3D) curve.

A problem one typically has to face when dealing with such *functional objects* is to find ways to compare them. This comparison problem is most of the time posed as a *matching problem*, which may be described as "finding similar structures appearing at similar places (or similar times)." To be more explicit, given two "objects" θ and θ', expressed as functions defined on the same interval I, the issue is to find, for each $x \in I$, some $x' \in I$ such that $x \simeq x'$ and $\theta(x) \simeq \theta'(x')$. The matching is *consistent* if the correspondence $x \mapsto x'$ is one-to-one, i.e., there cannot be two distinct locations on θ which are associated to the same location on θ'; it is *complete* if each location in θ is matched to some location in θ', and *bicomplete* if, in addition, each location in θ' is matched to some location in θ. Consistent bicomplete matchings thus can be represented by bijections $\phi : I \to I$, and the matching problem can be formulated as finding such a ϕ such that $\phi \simeq \mathrm{id}$ (where id is the identity function $x \mapsto x$) and $\theta \simeq \theta' \circ \phi$ (in this last sentence, $\simeq$ must be understood as "as close as possible to").

A common approach to realize this program is to minimize some functional $L_{\theta,\theta'}(\phi)$ which is small when the requirements above are satisfied. One simple example is (letting $\dot{\phi} = \frac{d\phi}{dx}$)

$$L_{\theta,\theta'}(\phi) = \int_I (\dot{\phi} - 1)^2 dx + \lambda \int_I (\theta(x) - \theta' \circ \phi(x))^2 dx, \tag{1.1}$$

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and many functionals which are used in the literature fall into this category, with some variations. One of the drawbacks in this formulation is the lack of symmetry in θ and θ'. In general, matching θ to θ' or θ' to θ would yield distinct results. This is undesirable, unless there is some reason to privilege one object to the other, and symmetrical matching seems appealing in many contexts. A sufficient condition yielding symmetrical matching would be

$$L_{\theta,\theta'}(\phi) = L_{\theta',\theta}(\phi^{-1}),$$

which is true for functionals of the kind (see section 2)

$$L_{\theta,\theta'}(\phi) = \int_I F(\dot\phi, \theta, \theta' \circ \phi) dx \tag{1.2}$$

with $\xi F(1/\xi, v, u) = F(\xi, u, v)$.

One way to modify (1.1) in order to put it into the above form could be to set

$$F(\xi, u, v) = (\sqrt{\xi} - 1)^2 + \lambda\sqrt{\xi}(u - v)^2.$$

Note that, denoting by $|I|$ the length of the interval I, one has, in this case, (because ϕ is increasing from I onto I)

$$\int_I F(\dot\phi, \theta(x), \theta' \circ \phi(x))dx = 2|I| - \int_I \sqrt{\dot\phi}\left(2 - \lambda(\theta(x) - \theta' \circ \phi(x))^2\right) dx,$$

and the problem is equivalent to *maximizing* $\int_I \sqrt{\dot\phi}\left(2 - \lambda(\theta(x) - \theta' \circ \phi(x))^2\right) dx$.

The problem which has motivated this paper is the functional which has been designed in [9], [10]. In this case, F is given by

$$F(\xi, u, v) = \sqrt{\xi}\left|\cos\frac{u - v}{2}\right|. \tag{1.3}$$

The functional can be used to compare and match plane curves, θ being in this case the angle between the tangent and some reference axis. It has been shown that the minimum, over ϕ, of $\arccos L_{\theta,\theta'}(\phi)$ provides a distance between plane curves seen up to translation and scaling. (That is, it is not only symmetrical but also satisfies the triangular inequality.) In fact, this function F comes in a very natural way from the computation of paths of minimal energy in the space of plane curves.

In the present paper, we only consider the case when F can be written as $\sqrt{\xi}G(u, v)$, and one tries to maximize

$$\int_0^1 \sqrt{\dot\phi}G(\theta, \theta' \circ \phi)dx.$$

Some discussion on how this formulation can be seen as a consequence of some simple assumptions on the matching will be provided in section 2. We shall then study the existence and the properties of solutions of this type of variational problem. More precisely, we shall ask whether there exists some optimal matchings ϕ, *which are bijective*. When such a ϕ exists, we then want to discuss on its smoothness properties, and check, in particular, that the normalization by $\sqrt{\dot\phi}$ has not harmed too much of the smoothing properties of the initial formulation (1.1).

If we forget about θ and θ', we must thus deal with variational problems which fit into the following framework. Without loss of generality, we take $I = [0,1]$ for the rest of the paper and we introduce some notations.

Notation 1. Let Hom^+ be the set of increasing homeomorphisms on $[0,1]$, i.e., the set of continuous strictly increasing functions $\phi : [0,1] \to [0,1]$ such that $\phi(0) = 0$, $\phi(1) = 1$.

Since $\phi \in \mathrm{Hom}^+$ is continuous and increasing, ϕ is differentiable almost everywhere (a.e.). This derivative is denoted $\dot{\phi}$.

Given a measurable function $f : [0,1] \times [0,1] \to \mathbb{R}_+$, we define, for $\phi \in \mathrm{Hom}^+$,

$$U_f(\phi) = \int_0^1 \sqrt{\dot{\phi}} f(\phi(x), x) dx \,.$$

We also let $\tilde{f}(x,y) = f(y,x)$.

Most of the paper (sections 4, 5, and 6) will be devoted to proving the results which are stated in section 3. Some auxiliary results will also be given in section 7. We start with a discussion on which general form a matching function $F(\xi, u, v)$ may assume when some simple invariance properties are required.

2. Invariance properties of the matching. Let a function F be defined on $[0, +\infty[\times \mathbb{R}^d \times \mathbb{R}^d$, and specify the problem of optimal matching between two functions θ and θ', defined on $[0,1]$, and with values in $\mathbb{R}^d$ through the functional, defined for all ϕ, which are increasing diffeomorphisms of $[0,1]$,

$$L_{\theta,\theta'}(\phi) = \int_0^1 F(\dot{\phi}(x), \theta(x), \theta' \circ \phi(x)) dx.$$

As said before, one desirable property is symmetry: for any functions θ and θ', we want that

$$\phi = \mathrm{argmax} L_{\theta,\theta'} \Leftrightarrow \phi^{-1} = \mathrm{argmax} L_{\theta',\theta}.$$

Since

$$L_{\theta',\theta}(\phi^{-1}) = \int_0^1 \dot{\phi}(x) F\left(\frac{1}{\dot{\phi}(x)}, \theta' \circ \phi(c), \theta(x)\right) dx,$$

a sufficient condition for symmetry is the following.

[C1] For all $(\xi, u, v) \in [0, +\infty[\times \mathbb{R}^d \times \mathbb{R}^d$, one has $F(\xi, u, v) = \xi F(1/\xi, v, u)$.

Since we maximize $L_{\theta,\theta'}$, one should build F with suitable properties with respect to maximization. One such property is that F should be concave with respect to its first variable ξ. (To understand why concavity in the first variable is essential for this kind of problem, one can refer to [2].) This is stated in condition [C2].

[C2] For all u, v, $\xi \mapsto F(\xi, u, v)$ is concave on $[0, +\infty[$.

It is important to notice that this concavity assumption is consistent with the symmetry [C1] in the following sense. If [C1] is not true, it is natural to try to symmetrize F by replacing it by

$$F^s(\xi, u, v) = F(\xi, u, v) + \xi F\left(\frac{1}{\xi}, v, u\right).$$

It is easily shown, then, that condition [C2] is true for F^s as soon as it was originally true for F.

Another natural condition for the functional is that, when comparing a function θ with itself, the optimal ϕ is $\phi = \mathrm{id}$. In other terms, one should have, for all functions θ and all diffeomorphisms ϕ,

$$\int_0^1 F(\dot{\phi}, \theta, \theta \circ \phi)dx \le \int_0^1 F(1, \theta, \theta)dx.$$

A sufficient condition for this can be that, for all ξ, u, v, $F(\xi, u, v) \le F(1, u, u)$. If one takes into account the constraint $\int_0^1 \dot{\phi} = 1$, this can be weakened into the following condition (which can be shown to be necessary and sufficient [8]).

[C3] There exists a measurable function $\lambda : \mathbb{R}^d \to \mathbb{R}$ such that, for all $\xi > 0, u, v \in \mathbb{R}^d$,

$$F(\xi, u, v) \le F(1, u, u) + \lambda(v)\xi - \lambda(u).$$

Indeed, assuming [C3], we have

$$\int_0^1 F(\dot{\phi}, \theta, \theta \circ \phi)dx \le \int_0^1 F(1, \theta, \theta)dx + \int_0^1 \dot{\phi}\lambda \circ \theta \circ \phi dx - \int_0^1 \lambda \circ \theta dx,$$

and the last two integrals are equal by a change of variables.

Additional constraints may come from invariance properties which may be imposed on the matching. The first invariance property we consider will be called "focus invariance." Consider θ and θ' as signals, defined on $[0, 1]$, and assume that they have been matched by some function ϕ^*. Let $[a, b]$ be a subinterval of $[0, 1]$, and set $[a', b'] = [\phi^*(a), \phi^*(b)]$. We want to refocus the matching on these intervals. For this, we can rescale the functions θ and θ' on these intervals to get new signals defined on $[0, 1]$, and match the new signals. The question which arises then is whether this new matching is consistent with the one which has been obtained initially.

Let us be more precise. To rescale θ (resp., θ'), we define $\theta_{ab}(x) = \theta(a + (b - a)x)$ (resp., $\theta'_{a'b'}(x) = \theta'(a' + (b' - a')x)$), $x \in [0, 1]$. Comparing these signals with the functional F yields an optimal matching which, if it exists, maximizes

$$\int_0^1 F(\dot{\phi}(x), \theta_{a,b}(x), \theta'_{a',b'} \circ \phi(x))dx. \tag{2.1}$$

The optimal matching between the initial functions θ and θ' clearly maximizes

$$\int_a^b F(\dot{\phi}(y), \theta(y), \theta' \circ \phi(y))dy$$

with the constraints $\phi(a) = a'$ and $\phi(b) = b'$. Making the change of variables $y = a + (b - a)x$ and setting $\psi(x) = (\phi(y) - a')/(b' - a')$, this integral can be written as

$$(b - a) \int_0^1 F(\lambda\dot{\psi}(x), \theta_{a,b}(x), \theta'_{a',b'} \circ \psi(x))dx \tag{2.2}$$

with $\lambda = \frac{b'-a'}{b-a}$. We say that F satisfies a focus invariance propertyies if, for any θ and θ', the maximizer of (2.1) is the same as the maximizer of (2.2). One possible condition ensuring such a property is that F is itself (relatively) invariant under the transformation $(\xi, u, v) = (\lambda\xi, u, v)$.

[Focus] For some $\alpha > 0$, for all $\xi > 0$, $u, v \in \mathbb{R}^d$, $F(\lambda\xi, u, v) = \lambda^\alpha F(\xi, u, v)$.

This condition trivially implies that $F(\xi, u, v) = \xi^\alpha F(1, u, v)$. If condition [C1] is imposed, one sees that α has to be equal to $1/2$. We thus get the following.

The only matching functionals which satisfy [C1] *and* [Focus] *take the form*

$$F(\xi, u, v) = \sqrt{\xi} F_1(u, v) \tag{2.3}$$

with $F_1(u, v) = F_1(v, u)$.

These functionals satisfy [C2] as soon as $F_1 \geq 0$. For [C3], we must have, for all $u, v \in \mathbb{R}^d$,

$$F_1(u, v) \leq \sqrt{F_1(u, u) F_1(v, v)}. \tag{2.4}$$

Indeed, assuming [C3], we must have, for some function λ,

$$F_1(u, u) - \lambda(u) = \max_{v, \xi} \sqrt{\xi} F_1(u, v) - \lambda(v)\xi.$$

For a fixed v, $\sqrt{\xi} F_1(u, v) - \lambda(v)\xi$ has a finite maximum if $\lambda(v) > 0$, or if $F_1(u, v) = \lambda(v) = 0$. In the first case, the maximum is given by

$$\frac{F_1(u, v)^2}{4\lambda(v)}.$$

This implies that

$$F_1(u, u) - \lambda(u) = \max_v \frac{F_1(u, v)^2}{4\lambda(v)} \tag{2.5}$$

with the convention $0/0 = 0$. In particular, taking $v = u$, one has

$$F_1(u, u)^2 - 4\lambda(u) F_1(u, u) + 4\lambda(u)^2 \leq 0,$$

which is possible only if $F_1(u, u) = 2\lambda(u)$. Given this fact, (2.5) clearly implies (2.4). We thus have the following.

The only matching functionals which satisfy [C1]–[C3] *and* [Focus] *take the form*

$$F(\xi, u, v) = \sqrt{\xi} F_1(u, v) \tag{2.6}$$

with $F_1(u, v) = F_1(v, u)$, $F_1(u, v) \geq 0$, *and* $F_1(u, v) \leq \sqrt{F_1(u, u) F_1(v, v)}$.

The functional in (1.3) satisfies this property.

One must note, however, that focus invariance under the above form is not a suitable constraint for every matching problem. Let us restrict our attention to the comparison of plane curves, which has initially motivated the present paper. In this case, the functions θ are typically geometrical features computed along the curve and expressed as functions of the (euclidean) arc-length. In such a context, focusing should rather be interpreted from a geometrical point of view, as rescaling (a portion of) a plane curve to length 1. But, in this case, applying such a scale change may have some impact not only on the variable x (which here represents the length), but also on the *values* of the geometric features θ. In example (1.3), the geometric features were the orientation of the tangents, which are not affected by scale change, so that focus invariance is in this case equivalent to geometric scale invariance. Letting κ be the curvature computed along the curve, the same invariance would be true if we had taken $\theta = \kappa'/\kappa^2$ (which is the "curvature" which characterizes curves up to similitudes). But if we had chosen to compare precisely euclidean curvatures, the invariance constraints on the matching would be different. Since curvatures are scaled by λ^{-1} when a curve is scaled by λ, the correct condition should be (instead of [Focus])

$$F(\lambda\xi, \lambda u, v) = \lambda^{\alpha} F(\xi, u, v).$$

This comes from rescaling only the first curve. Rescaling the second curve yields

$$F(\lambda\xi, u, v/\lambda) = \lambda^{\beta} F(\xi, u, v).$$

Note that, if the symmetry condition is valid, we must have $\beta = 1 - \alpha$, which we assume hereafter.

One can solve this identity and compute all the (continuously differentiable) functions which satisfy it. This yields functions F of the kind

$$F(\xi, u, v) = H\left(\xi \frac{v}{u}\right) u^{\alpha} v^{\alpha - 1}.$$

Note that, since F should be concave as a function of ξ, H itself should be concave. The symmetry condition is ensured as soon as $xH(1/x) = H(x)$ for all x. One may set

$$F(\xi, u, v) = -|\xi v - u|,$$

which satisfies [C1]–[C3].

Many variations can be done on these computations. Inspiration on how devising functionals which satisfy given criteria of invariance can be obtained from the first chapters of [4].

We now return to our original problem, which contains, according to the above terminology, symmetrical, focus invariant matchings.

3. Main results.

3.1. Existence results.

Notation 2. Let a and b be two points in $[0,1]^2$ such that $a = (a_1, a_2)$ and $b = (b_1, b_2)$. We denote by $[a, b]$ the closed segment from a to b, i.e., the set of points $a + t(b - a)$, for $0 \leq t \leq 1$, and by $]a, b[$ the open segment from a to b defined by $]a, b[= [a, b] \setminus \{a, b\}$. Moreover, we say that such a segment is horizontal (resp., vertical) if $a_2 = b_2$ (resp., $a_1 = b_1$).

Notation 3. Denote by Δ_f the integral $\Delta_f = \int_0^1 f(x, x)dx$.

Let Ω_f be the set

$$\Omega_f = \left\{ (y, x) \in [0,1]^2 \mid |x - y| \leq \sqrt{1 - \left(\frac{\Delta_f}{\|f\|_\infty}\right)^2} \right\}.$$

THEOREM 3.1. *Assume that $f \geq 0$ is bounded and satisfies conditions* [H1] *and* [H2].

[H1] *There exists a finite family of closed segments $([a_j, b_j])_{j \in J}$ such that each of them is horizontal or vertical and f is continuous on $[0,1]^2 \setminus F$, where $F = \bigcup_{j \in J} [a_j, b_j]$.*

[H2] *Let f_s be defined by*

$$f_s(x) = \lim_{\delta \to 0} \left(\inf\{ f(u) \mid u \in [0,1]^2 \setminus F, \ |u - x| < \delta \} \right).$$

There does not exist any nonempty open vertical or horizontal segment $]a, b[$ *such that* $]a, b[\subset \Omega_f$ *and* f_s *vanishes on* $]a, b[$.

Then there exists $\phi^* \in Hom^+$ *such that* $U_f(\phi^*) = \max\{U_f(\phi), \phi \in Hom^+\}$*. Moreover, if* ϕ *is a maximizer of* U_f*, one has, for all* $x \in [0, 1]$*,* $(\phi(x), x) \in \Omega_f$*.*

3.2. Regularity of the optimal matching. We now give some conditions under which the optimal matching satisfies some smoothness properties.

DEFINITION 3.2. *We say that* $f : [0, 1]^2 \to \mathbb{R}$ *is Hölder continuous at* (y, x) *if there exist* $\alpha > 0$ *and* $C > 0$ *such that*

$$|f(y', x') - f(y, x)| \leq C \max(|y' - y|^\alpha, |x' - x|^\alpha) \tag{3.1}$$

for any $(y', x') \in [0, 1]^2$*.*

We say that f *is locally uniformly Hölder continuous at* (y_0, x_0) *if there exists a neighborhood* V *of* (y_0, x_0) *such that* f *is Hölder continuous at all* $(y, x) \in V$*, with constants* C *and* α*, which are uniform over* V*.*

THEOREM 3.3. *Let* f *be a nonnegative real-valued measurable function on* $[0, 1]^2$ *and assume that* U_f *reaches its maximal value on* Hom^+ *at* ϕ^**. Then for any* $x_0 \in [0, 1]$*, if* $f(\phi(x_0), x_0) > 0$ *and if* f *is Hölder continuous at* $(\phi(x_0), x_0)$*, then* ϕ^* *is differentiable at* x_0 *with strictly positive derivative.*

Moreover, if f *is locally uniformly Hölder continuous, then* $\dot{\phi}^*$ *is continuous in a neighborhood of* x_0*.*

THEOREM 3.4. *Assume that* f *is continuously differentiable in both variables. Let* $\phi \in Hom^+$ *be such that* $U_f(\phi) = \max\{U_f(\psi) \,|\, \psi \in Hom^+\}$ *and that, for all* $x \in [0, 1]$*, one has* $f(\phi(x), x) > 0$*. Then,* ϕ *is twice continuously differentiable.*

4. Remarks.

4.1. Positivity of f. It is necessary to control the vanishing sets of the function f (as in condition [H2]) to obtain a homeomorphism. One simple example is when f vanishes on a square $S =]\frac{1}{2} - a, \frac{1}{2} + a[^2 \subset [0, 1]^2$, and $f = 1$ outside. (Such an f satisfies condition [H1].) Using the fact that (as a consequence of Lemma 5.16 below), ϕ must be linear on any section which does not encounter S, it is not very difficult to prove that the maximum is attained for ϕ which is *discontinuous* at $x = a + 1/2$, more precisely such that $\phi(x) = \frac{1}{2} - a$ and $\phi(x + 0) = \frac{1}{2} + a$.

4.2. Piecewise constant functions. A particular and important case in which condition [H1] is valid is the case of piecewise constant functions f. We state this as a corollary.

COROLLARY 4.1. *Assume that there exists* $0 = x_0 < x_1 < \cdots < x_m = 1$ *(resp.,* $0 = x'_0 < x'_1 < \cdots < x'_n = 1$*) and constants* $f_{kl} > 0$*,* $1 \leq k \leq m$*,* $1 \leq l \leq n$ *such that*

$$f(x, x') = \sum_{k=1}^{m} \sum_{l=1}^{n} f_{kl} \mathbf{1}_{[x_{k-1}, x_k[\times[x'_{l-1}, x'_l[}(x, x').$$

Then, there exists $\phi^* \in Hom^+$*, which is piecewise linear such that*

$$U_f(\phi^*) = \max\{U_f(\phi) \,|\, \phi \in Hom^+\}.$$

Remark. In fact, one needs to assume only that $f_{kl} > 0$ for k and l such that $[x_{k-1}, x_k[\times[x'_{l-1}, x'_l[$ is sufficiently close from the diagonal of the unit square, i.e., intersects the set Ω_f.

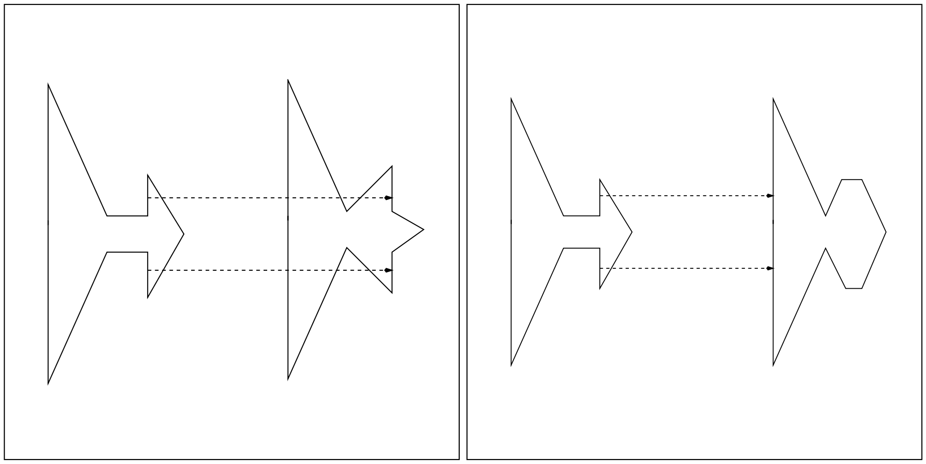

FIG. 4.1. *Matching of flat sections. Left: Situation in which* [H2] *is not true. A point on the first curve has a whole interval of possible matches on the second one with an opposite orientation of the tangent. Right: In this situation, there still exist portions in the second curve with opposite tangents, but because their respectve arc-lengths are far apart, they can be ignored (using the set* Ω_f*):* [H2] *is true.*

Proof. We need to show only that the optimal ϕ is piecewise linear. But we have

$$U_f(\phi) = \sum_{k=1}^{m}\sum_{l=1}^{n} f_{kl} \int_{x_{k-1}}^{x_k} \sqrt{\dot\phi(x)}\mathbf{1}_{[x_{l-1},x_l[}(\phi(x))dx.$$

Let $y_k = \phi(x_k)$, $k = 0, \dots, m$. Let $y'_l = \phi^{-1}(x'_l)$, $l = 0, \dots, n$. We have, writing $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$,

$$U_f(\phi) = \sum_{k=1}^{m}\sum_{l=1}^{n} f_{kl} \int_{x_{k-1}\vee y'_{l-1}}^{x_k\wedge y'_l} \sqrt{\dot\phi(x)}dx,$$

and Lemma 5.16 yields the result. ◻

4.3. Application to optimal matching of functions. Let us see what conditions [H1] and [H2] mean when f is of the kind

$$f(y, x) = F_1(\theta(y), \theta'(x)) .$$

One of the examples we have in mind is the case when $\theta(y)$ and $\theta'(y)$ take values in $[0, 2\pi[$, and $F_1(u, v) = |\cos[(u - v)/2]|$. In this case, these functions, θ and θ', correspond to *rotation angles* of the unitary tangents to some plane curves, and matching is used to compare shapes on the basis of their silhouettes.

In this particular case, F_1 is continuous, but not continuously differentiable. It is, however, smooth enough to fit into the regularity condition [H1], so that the true constraint is on θ and θ'. Note that f is discontinuous on a horizontal (resp., vertical) segment as soon as θ (resp., θ') is discontinuous at the position of the segment in the horizontal (resp., vertical) axis. Thus [H1] implies that θ and θ' are continuous except at a finite number of points. Points of discontinuity of θ and θ' are angular points for the plane curves they represent; thus condition [H1] implies that one can safely perform a matching between shapes having a finite number of angular points, which is the case of most of the objects which can be observed in a standard environment. Note that, in this case, piecewise constant f corresponds to polygonal shapes, which is also an important example to deal with.

Condition [H2] essentially means that one cannot have intervals on which, for a given x_0, $F(\theta(.), \theta'(x_0)) = 0$. In the case of curve matching, when θ is an angle, and formula (1.3) is used, this means that one of the curves cannot have a flat portion which may be matched to a point of the other curve with opposite tangent. (Note that one can restrict this condition to points which are located at close enough positions on both curves; see Figure 4.1 for an illustration.) In particular, the condition is always true if the compared curves contain no flat sections.

5. Proof of the existence.

5.1. Sketch of the proof. In the next section, we will introduce a compact set $\mathcal{D}^*$, containing Hom^+, and extend the functional U_f to this space. We first prove the existence of the maximum for this extended functional through the following proposition.

PROPOSITION 5.1. *If f satisfies condition* [H1]*, then U_f is upper-semicontinuous on $\mathcal{D}^*$. Since $\mathcal{D}^*$ is compact, there exists $\phi^* \in \mathcal{D}^*$ such that*

$$U_f(\phi^*) = \sup\{U_f(\psi) \,|\, \psi \in \mathcal{D}^*\}.$$

Moreover, if ϕ is a maximizer of U_f, one has, for all $x \in [0,1]$, $(\phi(x), x) \in \Omega_f$.

Theorem 3.1 will then be a consequence of the following proposition.

PROPOSITION 5.2. *If f satisfies condition* [H2] *in Theorem* 3.1 *and $\phi^* \in \mathcal{D}^*$ is such that*

$$U_f(\phi^*) = \sup\{U_f(\psi) \,|\, \psi \in \mathcal{D}^*\},$$

then $\phi \in Hom^+$.

Before proving these propositions, we introduce $\mathcal{D}^*$.

5.2. The set $\mathcal{D}^*$.

5.2.1. Definition. Let $\mathcal{M}_1$ be the set of the positive Radon measures μ on $[0,1]$ such that $\mu([0,1]) = 1$. Let $\mathcal{M}$ be the set of measures on $[0,1]$ such that $\mu([0,1]) \leq 1$. We let $\mathcal{D}^*$ be the set of all ϕ which can be written as

$$\phi(s) = \mu([0,s[)$$

for some $\mu \in \mathcal{M}_1$. Such ϕ are nondecreasing, left continuous, and satisfy $\phi(0) = 0$ and $\phi(1) \leq 1$. Conversely, any ϕ satisfying these properties is in $\mathcal{D}^*$, and the associated measure is unique and will be denoted by μ_ϕ. Note that $\mu_\phi(\{1\}) = 1 - \phi(1)$ for all $x \in [0,1[$, $\mu_\phi(\{x\}) = \phi(x+0) - \phi(x)$, where $\phi(x+0)$ is the right limit of ϕ at x.

Any $\mu \in \mathcal{M}_1$ can be written in a unique way under the form

$$\mu = \omega dx + \nu,$$

where dx is the Lebesgue measure on $[0,1]$, ω is a measurable, nonnegative, function on $[0,1]$, and ν is singular. (There exists a set E of Lebesgue measure 0 such that $\nu(A) = \nu(A \cap E)$ for every Borel set $A \subset [0,1]$.) For $\phi \in \mathcal{D}^*$, we take the notation

$$d\mu_\phi = \omega_\phi ds + d\nu_\phi.$$

DEFINITION 5.3. *For a function $\phi \in [0,1]$, we let $\dot\phi(x)$ be the limit when $\epsilon \to 0$ of $\frac{\phi(x+\epsilon)-\phi(x)}{\epsilon}$ when this limit exists and $\dot\phi(x) = 0$ otherwise. If $\phi \in \mathcal{D}^*$, one has $\dot\phi = \omega_\phi$ a.e.* [7, Theorem 8.18].

Following [6], we extend the functional U_f to $\mathcal{D}^*$ by letting

$$U_f(\phi) = \int_0^1 \sqrt{\dot\phi(x)} f(\phi(x), x) ds\,.$$

We also denote by $\mathcal{D}^*_+$ the set of functions $\phi \in \mathcal{D}^*$ for which $\int_a^b \dot\phi > 0$ for any $0 \le a < b \le 1$. We have $\phi \in \mathrm{Hom}^+$ if $\phi \in \mathcal{D}^*_+$, and ν_ϕ is diffuse, i.e., $\nu_\phi(\{x\}) = 0$ for any $x \in [0,1]$. (Note that this is not a necessary condition: there exists functions in Hom^+ such that $\dot\phi = 0$ a.e. See [3, example 18–8] .

5.2.2. Weak convergence in $\mathcal{D}^*$.

Measures $\mu_n \in \mathcal{M}, n \ge 0$ are said to converge for the weak*-topology to a limit μ if, for every continuous function F on $[0,1]$, one has

$$\lim_{n\to\infty} \int_0^1 F d\mu_n = \int_0^1 F d\mu.$$

Since this is the only kind of convergence we use on $\mathcal{M}$ and $\mathcal{M}_1$, the statement "μ_n converges to μ" will always mean convergence in the weak*-topology. We say that $\phi_n \in \mathcal{D}^*$ weakly converges to $\phi \in \mathcal{D}^*$ if μ_{ϕ_n} converges to μ_ϕ.

We list some results related to this convergence.

PROPOSITION 5.4 (see [5]). *The sets $\mathcal{M}$ and $\mathcal{M}_1$ are compact for the weak*-topology.*

If ϕ_n weakly converges to ϕ, then, for all $x \in [0,1]$ such that ϕ is continuous at x, one has $\phi_n(x) \to \phi(x)$.

Note that, since ϕ is increasing, its discontinuity set is at most countable.

PROPOSITION 5.5. *Let ϕ_n be a sequence in $\mathcal{D}^*$, such that $\dot\phi_n dx$ and ν_{ϕ_n} both converge in $\mathcal{M}$, respectively, to $\alpha dx + \rho$ and $\beta dx + \tau$. Then μ_{ϕ_n} converges to $\mu \in \mathcal{M}_1$ such that $\mu = (\alpha+\beta)dx + (\rho+\tau)$.*

This is obvious. Note that, by compactness of $\mathcal{M}$, from any sequence ϕ_n one can extract a subsequence such that both $\dot\phi_n dx$ and ν_{ϕ_n} converge.

We introduce, here and for what follows, a mollifier g, i.e., an infinitely differentiable mapping $g : \mathbb{R} \to \mathbb{R}$, with compact support included in $]-1,1[$, such that $\int_{-1}^1 g(x)dx = 1$. For $\epsilon > 0$, we let $g_\epsilon(x) = g(x/\epsilon)/\epsilon$. One has the following lemma.

LEMMA 5.6 (see [6]). *Let $\mu_n = \omega_n dx$ be a sequence of absolutely continuous measures in $\mathcal{M}$ which converges to $\alpha dx + \rho$. Then, for any $\epsilon > 0$, one has*

$$\lim_{n\to\infty} \int_0^1 |\omega_n * g_\epsilon - \alpha * g_\epsilon - \rho * g_\epsilon|\, dx = 0.$$

Here, $\mu * g_\epsilon$ denotes (as usually) the convolution of μ by g_ϵ,

$$\mu * g_\epsilon(x) = \int_0^1 g_\epsilon(x-y) d\mu(y)\,.$$

5.2.3. A symmetry property of U_f. For $\phi \in \mathcal{D}^*$, we define, for $y \in [0,1]$,

$$\phi^-(y) = \sup\{x \in [0,1] \,|\, \phi(x) < y\},$$

with the convention that $\sup\emptyset = 0$.

Our purpose is to prove the following proposition (recall that we have denoted $\tilde f(x,y) = f(y,x)$).

PROPOSITION 5.7. *For all* $\phi \in \mathcal{D}^*$*, one has*

$$U_f(\phi) = U_{\tilde f}(\phi^-).$$

The proof will be carried on with several lemmas.

LEMMA 5.8. *Let* $\phi \in \mathcal{D}^*$*.*

1. *We have* $\phi^- \in \mathcal{D}^*$ *and* $(\phi^-)^- = \phi$*.*
2. *For any* $x \in [0,1]$*,*

$$\phi^-(\phi(x)) \le x \le \phi^-(\phi(x)+0) \tag{5.1}$$

so that $\phi^- \circ \phi(x) = x$ *as soon as* ϕ^- *is continuous at* $\phi(x)$*.*

3. *For any* $y \in [0,1]$*,*

$$\phi(\phi^-(y)) \le y \le \phi(\phi^-(y)+0) \tag{5.2}$$

so that $\phi \circ \phi^-(y) = y$ *as soon as* ϕ *is continuous at* $\phi^-(y)$*.*

Moreover, if $\phi \in \mathcal{D}^*_+$*, then* ϕ^- *is continuous.*

Proof. ϕ^- is nondecreasing, $\phi^-(0) = 0$, $\phi^-(1) \le 1$, and ϕ^- is left continuous, since

$$\phi^-(y) = \sup_{h>0}\left(\sup\{x, \phi(x) < y-h\}\right) = \sup_{h>0} \phi^-(y-h)$$

so that $\phi^- \in \mathcal{D}^*$. Now, let us show that for any $(x,y) \in [0,1]^2$,

$$\phi(x) > y \Rightarrow \phi^-(y) < x. \tag{5.3}$$

Indeed (assume $x \neq 0$—otherwise the result is trivial), if $\phi(x) > y$, since ϕ is left continuous, there exists $h > 0$ such that $\phi(x-h) > y$ so that $\phi^-(y) \le x - h < x$. Moreover, we deduce from the definition of ϕ^- that for any $x, y \in [0,1]$,

$$\phi^-(y) < x \Rightarrow \phi(x) \ge y. \tag{5.4}$$

Now, since $\phi(x) = \sup\{\ y \in [0,1] \mid y < \phi(x)\ \}$, using (5.3) and (5.4), we get

$$\phi(x) \le \sup\{y \in [0,1] |\ \phi^-(y) < x\} = (\phi^-)^-(x) \le \sup\{y \in [0,1] | \phi(x) \ge y\} = \phi(x)$$

so that 1 is proved.

From 1, we deduce that $2 \iff 3$ so that it is sufficient to prove 3. For any y, there exists an increasing sequence x_n which converges to $\phi^-(y)$ such that $\phi(x_n) < y$. Since ϕ is left continuous, this yields $\phi \circ \phi^-(y) \le y$. Moreover, for all $h > 0$, one has $\phi(\phi^-(y)+h) \ge y$ so that $y \le \phi(\phi^-(y)+0)$.

Now, assume that $\phi \in \mathcal{D}^*_+$ and assume that ϕ^- is discontinuous at $y_0 \in [0,1[$. Then $\phi^-(y_0+0) > \phi^-(y_0)$ so that $(\phi^-)^-$ has the constant value y_0 on $]\phi^-(y_0), \phi^-(y_0+0)]$. Since $(\phi^-)^- = \phi$, we get a contradiction with the fact that ϕ is strictly increasing. ∎

Note that ϕ is continuous at $\phi^-(y)$ if and only if $\mu_\phi(\{\phi^-(y)\}) = \nu_\phi(\{\phi^-(y)\}) = 0$.

LEMMA 5.9. *Let* $\phi \in \mathcal{D}^*$*. If* ϕ *is derivable at* x*, then*

$$\lim_{h\to 0} \frac{\phi(x+h+0) - \phi(x)}{h} = \dot\phi(x).$$

Proof. Let us show that, for any sequence x_n which converges to x, $x_n \neq x$, one has $\frac{\phi(x_n+0)-\phi(x)}{x_n-x} \to \dot\phi(x)$. Since ϕ is increasing, the limit is clearly larger than $\dot\phi(x)$. Letting $\epsilon_n = (x_n - x)^2 > 0$, we have

$$\frac{\phi(x_n+0)-\phi(x)}{x_n-x} \leq \frac{\phi(x_n+\epsilon_n)-\phi(x)}{x_n-x} = \frac{\phi(x_n+\epsilon_n)-\phi(x)}{x_n+\epsilon_n-x}(1+|x_n-x|),$$

and the last term converges to $\dot\phi(x)$. ∎

Define $P_\phi = \{ x \in [0,1] \mid \dot\phi(x) > 0 \}$ to be the set of locations where the derivative of ϕ exists and is strictly positive (see Definition 5.3). One has the following lemma.

LEMMA 5.10. *For any $x_0 \in [0,1]$, $x_0 \in P_\phi \iff \phi(x_0) \in P_{\phi^-}$. Hence,*

$$\dot\phi(x) = \frac{1}{\dot{\phi^-}(\phi(x))}\mathbf{1}_{P_{\phi^-}}(\phi(x))$$

(with the convention $0/0 = 0$).

Proof. ($\Leftarrow$) Assume that $\phi(x_0) \in P_{\phi^-}$; then ϕ is continuous at x_0. Indeed, if $\phi(x_0) < \phi(x_0+0)$, then ϕ^- is constant on $]\phi(x_0), \phi(x_0+0)[$ so that $\dot{\phi^-}(\phi(x_0)) = 0$ and $\phi(x_0) \notin P_{\phi^-}$ (which is a contradiction). Moreover, since ϕ^- is continuous at $\phi(x_0)$, we deduce from Lemma 5.8 that $\phi^-(y_0) = x_0$, where $y_0 = \phi(x_0)$. Now, noting that for any $h \in \mathbb{R}^*$ such that $x_0 + h \in [0,1]$, $\phi(x_0+h) \neq \phi(x_0)$ (otherwise ϕ^- should be discontinuous at $y_0 = \phi(x_0)$), we get using (5.1) and the fact that $\phi^-(\phi(x_0)) = x_0$

$$\frac{\phi(x_0+h)-\phi(x_0)}{\phi^-[\phi(x_0+h)+0]-\phi^-[\phi(x_0)]} \leq \frac{\phi(x_0+h)-\phi(x_0)}{h} \leq \frac{\phi(x_0+h)-\phi(x_0)}{\phi^-[\phi(x_0+h)]-\phi^-[\phi(x_0)]}.$$

Since ϕ is continuous at x_0, Lemma 5.9 applied to ϕ^- implies that $[\phi(x_0+h)-\phi(x_0)]/h$ converges to $(\dot{\phi^-} \circ \phi(x_0))^{-1} > 0$ when h tends to 0 so that $x_0 \in P_\phi$.

($\Rightarrow$) Now, assume that $x_0 \in P_\phi$. Then if $y_0 = \phi(x_0)$, ϕ^- is continuous at y_0. Indeed, if $\phi^-(y_0) < \phi^-(y_0+0)$, then since $\phi = (\phi^-)^-$, ϕ is constant on $]\phi^-(y_0), \phi^-(y_0+0)]$. However, from (5.1), we get that $x_0 \in [\phi^-(y_0), \phi^-(y_0+0)]$ so that $\dot\phi(x_0) = 0$ (which is a contradiction). Hence, ϕ^- is continuous at $y_0 = \phi(x_0)$ and $\phi^-(y_0) = x_0$. Now, using the part ($\Leftarrow$) for ϕ^-, we deduce that $y_0 = \phi^-(y_0) \in P_\phi = P_{(\phi^-)^-}$ implies $y_0 \in P_{\phi^-}$, i.e., $\phi(x_0) \in P_{\phi^-}$ so that the proof is finished. ∎

LEMMA 5.11. *If $\phi \in \mathcal{D}^*$ and g is a measurable function, one has, for all $x \in [0,1]$,*

$$\int_0^{\phi(x)} g \circ \phi^-(v)dv = \int_0^x g(u)\dot\phi(u)du + \int \mathbf{1}_{[0,x[}(u)g(u)d\nu_\phi(u). \tag{5.5}$$

Proof. This lemma can be proved first for $g = \mathbf{1}_{[0,b[}$ and extended to any g in a standard way. ∎

We can now prove Proposition 5.7. Applying Lemma 5.10 to ϕ^- instead of ϕ, we get

$$\begin{aligned} U_{\tilde f}(\phi^-) &= \int_0^1 \frac{\mathbf{1}_{P_\phi}(\phi^-(u))}{\sqrt{\dot\phi \circ \phi^-(u)}} f(u, \phi^-(u))du \\ &= \int_0^{\phi(1)} \frac{\mathbf{1}_{P_\phi}(\phi^-(u))}{\sqrt{\dot\phi \circ \phi^-(u)}} f(\phi \circ \phi^-(u), \phi^-(u))du. \end{aligned}$$

To justify this equality, we must show that the replacements of 1 by $\phi(1)$ and of u by $\phi \circ \phi^-(u)$ are valid. Assume that $\phi(1) < 1$. This implies that $\nu_\phi(\{1\}) > 0$ and thus that $1 \notin P_\phi$; for $u > \phi(1)$, one has $\phi^-(u) = 1$, so that $\mathbf{1}_{P_\phi}(\phi^-(u)) = 0$, which justifies the first replacement. For the second one, one has $\phi \circ \phi^-(x) \neq x$ only if ϕ is discontinuous at $\phi^-(x)$ and so not differentiable at $\phi^-(x)$, so that $\phi^-(x) \notin P_\phi$.

Now, Lemma 5.11 implies $U_{\tilde{f}}(\phi^-) = U_f(\phi)$ since ϕ is not derivable (hence, with our convention $\dot{\phi} = 0$) ν_ϕ a.e. [7, Theorem 8.11].

5.3. Proof of Proposition 5.1. We prove that U_f is upper-semicontinuous on $\mathcal{D}^*$. Let us consider the following lemma.

LEMMA 5.12. *Let f be a nonnegative function on $[0,1]^2$ which satisfies* [H1]. *Then, there exists a sequence $(f_n)_{n\geq 0}$ of continuous and nonnegative functions on $[0,1]^2$ such that for all $\phi \in \mathcal{D}^*$*

$$U_f(\phi) = \inf_{n\geq 0} U_{f_n}(\phi).$$

Proof. Let $F = \bigcup_{j\in J}[a_j, b_j]$ be the compact set defined in Theorem 3.1, let $M = \|f\|_\infty$, and consider the sequence $(f_n)_{n\geq 0}$ of nonnegative functions defined by

$$f_n(x) = (1 - \alpha_n(x))M + \alpha_n(x) f(x),$$

where $\alpha_n(x) = (Cd(x,F))^{1/n}$ and $d(x,F)$ is the usual distance from x to F and $0 < C < 1/\sqrt{2}$. One easily shows that f_n is continuous on $[0,1]$ and that $(f_n(x))_{n\geq 0}$ is a decreasing sequence converging to $f(x)$ for any $x \in [0,1]^2 \setminus F$.

Now consider $\phi \in \mathcal{D}^*$. By definition of the f_n's, $U_{f_n}(\phi)$ is a decreasing sequence. To show the result, it is sufficient to prove that

$$\sqrt{\dot{\phi}} f_n(\phi(x), x) \to \sqrt{\dot{\phi}} f(\phi(x), x) \text{ a.e.}$$

The result is obviously true for $x \in \{\, z \in [0,1] \mid \dot{\phi}(z) = 0 \text{ or } (\phi(z), z) \in [0,1]^2 \setminus F\}$. However, the set $\mathcal{F} = \{\, z \in [0,1] \mid \dot{\phi}(z) > 0,\ (\phi(z), z) \in F \,\}$ contains only isolated points, so that the result is proved: indeed, by contradiction assume that there exist $x \in \mathcal{F}$ and a sequence $(x_n)_{n\geq 0}$ of points of $\mathcal{F} \setminus \{x\}$ converging to x. Since F contains only a finite number of segments, there exists $j_0 \in J$ such that (up to the extraction of a subsequence) $x_n \in [a_{j_0}, b_{j_0}]$ for all $n \geq 0$. Moreover, since there exist n and n' such that $x_n \neq x_{n'}$, the segment $[a_{j_0}, b_{j_0}]$ is vertical so that $\phi(x_n)$ has a constant value and $\dot{\phi}(x) = 0$, which is a contradiction. □

Using Lemma 5.12, we deduce that if Theorem 3.1 is proved for nonnegative and continuous f, then, using the fact that the infimum of a family of upper-semicontinuous functions is upper-semicontinuous, we will get the result for any f nonnegative and satisfying condition [H1]. Hence, we can assume that f is continuous and nonnegative and prove that U_f is upper-semicontinuous.

For this, we consider a sequence $\phi_n \in \mathcal{D}^*$ such that ϕ_n weakly converges to ϕ in $\mathcal{D}^*$ (i.e., μ_{ϕ_n} converges to μ_ϕ in $\mathcal{M}$). Replacing, if needed, ϕ_n by a subsequence, we assume that both $\dot{\phi}_n dx$ and ν_{ϕ_n} converge in $\mathcal{M}$, respectively to $\alpha dx + \rho$ and $\beta dx + \tau$. By Proposition 5.5, ϕ_n weakly converges to $\phi \in \mathcal{D}^*$ with $\mu_\phi = (\alpha + \beta)dx + (\rho + \tau)$. We shall show that $\limsup U_f(\phi_n) \leq U_f(\phi)$. We have

$$U_f(\phi) - U_f(\phi_n) = \int_0^1 \left(\sqrt{\dot{\phi}}(x) - \sqrt{\dot{\phi}_n}(x)\right) f(\phi(x), x)dx + \int_0^1 \sqrt{\dot{\phi}_n}(x)(f(\phi(x), x) - f(\phi_n(x), x))dx.$$

Moreover,

$$\begin{aligned}
&\left|\int_0^1 \sqrt{\dot{\phi}_n}(x)(f(\phi(x),x) - f(\phi_n(x),x))dx\right| \\
&\leq \int_0^1 \sqrt{\dot{\phi}_n}(x)\left|(f(\phi(x),x) - f(\phi_n(x),x))\right| dx \\
&\leq \left[\int_0^1 \dot{\phi}_n(x)dx\right]^{1/2} \left[\int_0^1 \left|(f(\phi(x),x) - f(\phi_n(x),x))\right|^2 dx\right]^{1/2} \\
&= \left[\int_0^1 \left|(f(\phi(x),x) - f(\phi_n(x),x))\right|^2 dx\right]^{1/2}.
\end{aligned}$$

This last integral tends to 0 by dominated convergence, since ϕ_n converges to ϕ a.e. (Proposition 5.4) and f is continuous. We thus have

$$\lim_{n\to\infty} \int_0^1 \sqrt{\dot{\phi}_n}(x)(f(\phi(x),x) - f(\phi_n(x),x))dx = 0. \tag{5.6}$$

We now study

$$\int_0^1 \left(\sqrt{\dot{\phi}}(x) - \sqrt{\dot{\phi}_n}(x)\right) f(\phi(x),x)dx.$$

We show that

$$\limsup \int_0^1 \sqrt{\dot{\phi}_n}(x) f(\phi(x),x)dx$$

is smaller than

$$\int_0^1 \sqrt{\alpha(x)} f(\phi(x),x)dx.$$

Since

$$\int_0^1 \sqrt{\dot{\phi}}(x) f(\phi(x),x)dx = \int_0^1 \sqrt{\alpha(x)+\beta(x)} f(\phi(x),x)dx,$$

this will prove Proposition 5.1.

We follow the method of [6], using the mollifier g_ϵ. We first prove the following lemmas.

Lemma 5.13. *For any $\epsilon > 0$, we have*

$$\lim_{n\to\infty} \int_0^1 \sqrt{\dot{\phi}_n * g_\epsilon(x)} f(\phi(x),x)dx = \int_0^1 \sqrt{\alpha * g_\epsilon(x) + \rho * g_\epsilon(x)} f(\phi(x),x)dx. \tag{5.7}$$

Proof. Indeed, we have

$$\begin{aligned}
&\int_0^1 \left|\sqrt{\dot{\phi}_n * g_\epsilon(x)} - \sqrt{\alpha * g_\epsilon(x) + \rho * g_\epsilon(x)}\right| f(\phi(x),x)dx \\
&\leq \|f\|_\infty \int_0^1 \left|\sqrt{\dot{\phi}_n * g_\epsilon(x)} - \sqrt{\alpha * g_\epsilon(x) + \rho * g_\epsilon(x)}\right| dx \\
&\leq \|f\|_\infty \left[\int_0^1 |\dot{\phi}_n * g_\epsilon(x) - (\alpha * g_\epsilon(x) + \rho * g_\epsilon(x))| dx\right]^{1/2} \\
&\leq \|f\|_\infty \left[\int_{-\infty}^{\infty} |\dot{\phi}_n * g_\epsilon(x) - (\alpha * g_\epsilon(x) + \rho * g_\epsilon(x))| dx\right]^{1/2},
\end{aligned}$$

which tends to 0 by Lemma 5.6. We have used the inequality $|\sqrt{a}-\sqrt{b}|^2 \leq |a-b|$, which is true for all $a, b \geq 0$. ∎

LEMMA 5.14. *We have*

$$\lim_{\epsilon\to 0}\sup_n \left|\int_0^1 \sqrt{\dot\phi_n} * g_\epsilon(x) f(\phi(x),x)dx - \int_0^1 \sqrt{\dot\phi_n} f(\phi(x),x)dx\right|.$$

Proof. For any $\eta > 0$ one can find a continuous function F_η on $[0,1]$ such that

$$\int_0^1 |F_\eta(x) - f(\phi(x),x)|^2 dx < \eta^2\,.$$

Fixing such an η, one has

$$\int_0^1 \sqrt{\dot\phi_n} * g_\epsilon(x)\,|f(\phi(x),x) - F_\eta(x)|\,dx \leq \eta\left[\int_0^1 \dot\phi_n * g_\epsilon(x)dx\right]^{1/2} \leq \eta,$$

where we have used the fact that, by Jensen's inequality, we have for all $n \geq 0$

$$\sqrt{\dot\phi_n} * g_\epsilon(x) \leq \sqrt{\dot\phi_n * g_\epsilon}.$$

Similarly,

$$\int_0^1 \sqrt{\dot\phi_n(x)}\,|f(\phi(x),x) - F_\eta(x)|\,dx \leq \eta.$$

We have

$$\begin{aligned}
&\left|\int_0^1 \sqrt{\dot\phi_n} * g_\epsilon(x) F_\eta(x)dx - \int_0^1 \sqrt{\dot\phi_n(y)} F_\eta(y)dy\right| \\
&\leq \int_{-\epsilon}^{1+\epsilon} dx \int_0^1 dy \sqrt{\dot\phi_n}(y) g_\epsilon(x-y)|F_\eta(x) - F_\eta(y)| \\
&\leq K_\eta(\epsilon)\int_0^1 \sqrt{\dot\phi_n}(y)dy \int_{-\epsilon}^{1+\epsilon} g_\epsilon(x-y)dx \\
&\leq K_\eta(\epsilon)\int_0^1 \sqrt{\dot\phi_n}(y)dy \leq K_\eta(\epsilon),
\end{aligned}$$

where $K_\eta(\epsilon) = \sup_{|y-x|\leq\epsilon}(|F_\eta(x) - F_\eta(y)|)$: we have used the fact that, for all y,

$$\int_{-\epsilon}^{1+\epsilon} g_\epsilon(x-y)dx \leq \int_{-\infty}^{\infty} g_\epsilon(x-y)dx = \int_{-\infty}^{\infty} g_\epsilon(x)dx = 1$$

and

$$\int_0^1 \sqrt{\dot\phi_n}(y)dy \leq 1.$$

Hence, we deduce that, for all n,

$$\left|\int_0^1 \left(\sqrt{\dot\phi_n} * g_\epsilon(x) - \sqrt{\dot\phi_n(x)}\right) f(\phi(x),x)dx\right| \leq 2\eta + K_\eta(\epsilon).$$

Since for $\eta > 0$, $K_\eta(\epsilon) \to 0$ when ϵ vanishes, we get the result. □

We end with the following lemma.

LEMMA 5.15. *We have*

$$\lim_{\epsilon\to 0}\int_0^1 \sqrt{\alpha * g_\epsilon(x) + \rho * g_\epsilon(x)} f(\phi(x), x)dx = \int_0^1 \sqrt{\alpha(x)} f(\phi(x), x)dx. \tag{5.8}$$

Proof. Indeed,

$$\begin{aligned}
&\left|\int_0^1 \left[\sqrt{\alpha * g_\epsilon(x) + \rho * g_\epsilon(x)} - \sqrt{\alpha(x)}\right] f(\phi(x), x)dx\right| \\
&\leq \left|\int_0^1 \left[\sqrt{\alpha * g_\epsilon(x) + \rho * g_\epsilon(x)} - \sqrt{\alpha * g_\epsilon(x)}\right] f(\phi(x), x)dx\right| \\
&\quad + \left|\int_0^1 \left[\sqrt{\alpha * g_\epsilon(x)} - \sqrt{\alpha(x)}\right] f(\phi(x), x)dx\right| \\
&\leq \|f\|_\infty \left|\int_0^1 \sqrt{\rho * g_\epsilon(x)}dx\right| + \|f\|_\infty \left|\int_0^1 \left[\sqrt{\alpha * g_\epsilon(x)} - \sqrt{\alpha(x)}\right] dx\right|.
\end{aligned}$$

One has $\rho * g_\epsilon(x) = \int_0^1 g_\epsilon(x-y)d\rho(y) \leq \frac{1}{\epsilon}\rho(]x-\epsilon, x+\epsilon[)$. Since ρ is singular, this upper-bound tends to 0 a.e. (cf., for example, [7, Theorem 8.6]). Thus

$$\begin{aligned}
\int_0^1 \sqrt{\rho * g_\epsilon(x)}dx &\leq \int_0^1 \sqrt{\rho * g_\epsilon(x)}\mathbf{1}_{\rho * g_\epsilon(x)\leq 1}dx + \int_0^1 \sqrt{\rho * g_\epsilon(x)}\mathbf{1}_{\rho * g_\epsilon(x)>1}dx \\
&\leq \int_0^1 \sqrt{\rho * g_\epsilon(x)}\mathbf{1}_{\rho * g_\epsilon(x)\leq 1}dx \\
&\quad + \left[\int_0^1 \rho * g_\epsilon(x)dx\right]^{1/2} \left[\int_0^1 \mathbf{1}_{\rho * g_\epsilon(x)>1}dx\right]^{1/2} \\
&\leq \int_0^1 \sqrt{\rho * g_\epsilon(x)}\mathbf{1}_{\rho * g_\epsilon(x)\leq 1}dx + \left[\int_0^1 \mathbf{1}_{\rho * g_\epsilon(x)>1}dx\right]^{1/2},
\end{aligned}$$

which tends to 0 by dominated convergence. We have used the fact that $\int_0^1 \rho * g_\epsilon(x)dx \leq 1$.

On the other hand,

$$\left|\int_0^1 \left[\sqrt{\alpha * g_\epsilon(x)} - \sqrt{\alpha(x)}\right] dx\right|^2 \leq \int_0^1 |\alpha * g_\epsilon(x) - \alpha(x)|dx,$$

which tends to 0 by [1, Theorem IV.22] . □

We can now end the proof of Proposition 5.1. For any $\eta > 0$, we deduce from Lemmas 5.15 and 5.14 that there exists $\epsilon > 0$ so that

$$\int_0^1 \sqrt{\alpha * g_\epsilon(x) + \rho * \epsilon(x)} f(\phi(x), x)dx \leq \int_0^1 \sqrt{\alpha(x)} f(\phi(x), x)dx + \eta,$$

and for all $n \geq 0$

$$\int_0^1 \sqrt{\dot{\phi}_n} f(\phi(x), x)dx \leq \int_0^1 \sqrt{\dot{\phi}_n * g_\epsilon(x)} f(\phi(x), x)dx + \eta.$$

Now, using Lemma 5.13, we deduce that for n sufficiently large, we have

$$\int_0^1 \sqrt{\dot{\phi}_n * g_\epsilon(x)} f(\phi(x), x) dx \leq \int_0^1 \sqrt{\alpha * g_\epsilon(x) + \rho * \epsilon(x)} f(\phi(x), x) dx + \eta.$$

Moreover, by Jensen's inequality, we have, for all n, $\sqrt{\dot{\phi}_n} * g_\epsilon(x) \leq \sqrt{\dot{\phi}_n * g_\epsilon}$ so that, using the previous inequalities, we get for sufficiently large n

$$\int_0^1 \sqrt{\dot{\phi}_n} f(\phi(x), x) dx \leq \int_0^1 \sqrt{\alpha(x)} f(\phi(x), x) dx + 3\eta.$$

Taking the lim sup and since η is arbitrary, we get the result.

We now prove the last statement of Proposition 5.1, that is, the fact that if ϕ is a miximizer of U_f, then, for all $x \in [0, 1]$, one has $(\phi(x), x) \in \Omega_f$. We start with a simple fact.

LEMMA 5.16. *Let* $[a, b] \subset [0, 1]$ *and* $[\tilde{a}, \tilde{b}] \subset [0, 1]$. *Then*

$$\max \left\{ \int_a^b \sqrt{\dot{\phi}(x)} dx \mid \phi \in \mathcal{D}^*, \phi(a) = \tilde{a}, \phi(b) = \tilde{b} \right\} = \sqrt{b-a}\sqrt{\tilde{b} - \tilde{a}},$$

and the maximum is attained for ϕ *linear between* a *and* b.

Proof. Indeed,

$$\left[\int_a^b \sqrt{\dot{\phi}} \right]^2 \leq (b-a) \int_a^b \dot{\phi} \leq (b-a)(\tilde{b} - \tilde{a})$$

with equality if ϕ is linear. □

LEMMA 5.17.

$$U_f(\phi) \leq \|f\|_\infty \sqrt{1 - \|\phi - \mathrm{id}\,\|_\infty^2}.$$

Proof. Take $x \in [0, 1]$ and set $M = |\phi(x) - x|$. Assume first that $\phi(x) = x + M$. Applying Lemma 5.16 between 0 and x and between x and 1, we get

$$U_f(\phi) \leq \|f\|_\infty \int_0^1 \sqrt{\dot{\phi}} dx \leq \|f\|_\infty \left(\sqrt{x}\sqrt{x + M} + \sqrt{1-x}\sqrt{1 - x - M} \right),$$

and elementary calculus yields that the right-hand side is always smaller than

$$\|f\|_\infty \sqrt{1 - M^2}\,.$$

The case $\phi(x) = x - M$ is handled similarly and yields the same upper-bound.

We thus have that, for all $x \in [0, 1]$,

$$U_f(\phi) \leq \|f\|_\infty \sqrt{1 - |\phi(x) - x|^2},$$

and taking the infimum of the upper-bound over all x yields the conclusion of the lemma. □

Now, if $U_f^* = \sup\{U_f(\phi), \phi \in \mathcal{D}^*\}$, we always have

$$U_f^* \geq U_f(\mathrm{id}\,) = \int_0^1 f(x, x) dx = \Delta_f\,.$$

Thus, if $U_f(\phi) = U_f^*$, we have

$$\Delta_f \leq U_f(\phi) \leq \|f\|_\infty \sqrt{1 - \|\phi - \mathrm{id}\,\|_\infty^2}\,;$$

that is,

$$\|\phi - \mathrm{id}\,\|_\infty \leq \sqrt{1 - \left(\frac{\Delta_f}{\|f\|_\infty}\right)^2}\,,$$

which concludes the proof of Proposition 5.1.

5.4. Proof of Proposition 5.2. Let $\phi \in \mathcal{D}^*$ such that $U_f(\phi)$ is maximal. We denote by m the Lebesgue's measure on $[0,1]$. Proposition 5.2 is an obvious consequence of the two following lemmas.

LEMMA 5.18. *For any* $0 \leq a < b \leq 1$, *we have* $\int_a^b \dot\phi(x)dx > 0$, *i.e.,* $\phi \in \mathcal{D}_+^*$.

Proof. Let us assume that Ω_f has a nonempty interior; that is, $\Delta_f < \|f\|_\infty$. (If this is not the case, Proposition 5.1 implies that the only maximizer is $\phi = \mathrm{id}$, which trivially belongs to $\mathcal{D}_+^*$.)

First, let us prove that $U_f(\phi) > 0$. Indeed, from condition [H2], we get that there exists a point (y_0, x_0) in the interior of $[0,1]^2$ such that $f_s(y_0, x_0) > 0$. Since f_s is lower-semicontinuous, f_s is strictly positive in a small neighborhood of (y_0, x_0). Now, define $\tilde\phi$ such that $\tilde\phi(0) = 0$, and $\tilde\phi(1) = 1$, $\tilde\phi(x_0) = y_0$, and $\tilde\phi$ is linear on $[0, x_0]$ and $[x_0, 1]$ (by Lemma 5.16). Since $\tilde\phi$ is strictly increasing on $[0,1]$, we deduce from [H1] that except possibly for a finite number of x, $(\phi(x), x) \notin F$ so that $f(\phi(x), x) = f_s(\phi(x), x)$. Hence, $U_f(\phi) \geq U_f(\tilde\phi) = U_{f_s}(\tilde\phi) > 0$.

Now assume that there exists $0 \leq a < b \leq 1$ such that $\int_a^b \dot\phi(x)dx = 0$. Let $a' = \inf\{z \in [0,a] \mid \int_z^b \dot\phi(x)dx = 0\ \}$ and $b' = \sup\{\ z \in [b,1] \mid \int_b^z \dot\phi(x)dx = 0\ \}$. We have $\int_{a'}^{b'} \dot\phi(x)dx = 0$, and since $U_f(\phi) > 0$, we have $a' > 0$ or $b' < 1$. Assume that $b' < 1$. (The case $a' > 0$ can be handled similarly.)

Now, for any $\eta > 0$ such that $b' + \eta \leq 1$, and any $\alpha \in]0,1[$, let us define

$$\omega^{\alpha,\eta}(x) = \dot\phi(x)\mathbf{1}_{x \notin [a', b'+\eta]} + (1-\alpha)\dot\phi(x)\mathbf{1}_{x \in]b', b'+\eta]} + \alpha \frac{K_\phi^\eta}{b' - a'}\mathbf{1}_{[a', b']},$$

where $K_\phi^\eta = \int_{a'}^{b'+\eta} \dot\phi(x)dx = \int_{b'}^{b'+\eta} \dot\phi(x)dx$. Let $\mu_{\alpha,\eta} = \omega^{\alpha,\eta} m + \nu_\phi$. One has $\mu_{\alpha,\eta} \in \mathcal{M}_1$, and, letting $\phi_{\alpha,\eta}(x) = \mu_{\alpha,\eta}([0,x[)$, one has

$$|\phi_{\alpha,\eta}(x) - \phi(x)| = \alpha \left| \int_0^x \left(\frac{K_\phi^\eta}{b' - a'}\mathbf{1}_{[a',b']} - \dot\phi \mathbf{1}_{]b', b'+\eta]} \right) \right| \leq \alpha \int_0^1 \dot\phi \leq \alpha,$$

so that $\|\phi_{\alpha,\eta} - \phi\|_\infty \leq \alpha$.

Let us show that $\phi(a') = \phi(b')$. Let R be the rectangle containing the points (y, x) such that $x \in]a', b'[$ and $y \in]\phi(a'), \phi(b')[$. If $\phi(a') < \phi(b')$, then this rectangle has a nonempty interior. Since $(\phi(a'), a') \in \Omega_f$ and $(\phi(b'), b') \in \Omega_f$, the intersection $R \cap \Omega_f$ also has a nonempty interior and, in particular, contains horizontal segments on which f cannot identically vanish. Thus, if $\phi(a') < \phi(b')$, there exist $x_0 \in]a', b'[$ and $y_0 \in\ ]\phi(a'), \phi(b')[$ such that $f_s(y_0, x_0) > 0$, and this in turn implies that f_s is strictly positive in a small neighborhood of (y_0, x_0). Now considering $\tilde\phi$ such that $\tilde\phi(x) = \phi(x)$ on $[0, a'] \cup [b', 1]$, $\tilde\phi(x_0) = y_0$, and $\tilde\phi$ is linear on $[a', x_0]$ and on $[x_0, b']$, we have (using the same argument as in the beginning of the proof) $U_f(\phi) < U_f(\tilde{\phi})$ with $\tilde{\phi} \in \mathcal{D}^*$, which is a contradiction.

Since $\phi(a') = \phi(b')$, the segment with end points $(\phi(a'), a')$ and $(\phi(b'), b')$ is vertical and clearly lies in Ω_f, so that by condition [H2] there exists $x_0 \in]a', b'[$ such that $f_s(\phi(x_0), x_0) > 0$. Using the fact that $\|\phi_{\alpha,\eta} - \phi\|_\infty \leq \alpha$, we deduce that there exist $\delta > 0$ and $c > 0$ such that for any sufficiently small α and any $x \in [0,1]$, except eventually a finite number (we use here the fact that $\phi_{\alpha,\eta}$ is strictly increasing on $[a', b']$ and condition [H1]), if $|x - x_0| < \delta$, we have $f(\phi_{\alpha,\eta}(x), x) = f_s(\phi_{\alpha,\eta}(x), x) \geq c$.

Now we have

$$\begin{aligned} U_f(\phi_{\alpha,\eta}) - U_f(\phi) = &\int_{a'}^{b'} \sqrt{\frac{\alpha K_\phi^\eta}{b'-a'}} f(\phi_{\alpha,\eta}(x), x)dx \\ &+ \int_{b'}^{b'+\eta} \left[\sqrt{(1-\alpha)\dot{\phi}(x)} f(\phi_{\alpha,\eta}(x), x) - \sqrt{\dot{\phi}(x)} f(\phi(x), x)\right] dx. \end{aligned}$$

However, for α sufficiently small,

$$\int_{a'}^{b'} \sqrt{\frac{\alpha K_\phi^\eta}{b'-a'}} f(\phi_{\alpha,\eta}(x), x)dx \geq \left(\frac{2c\delta}{\sqrt{b'-a'}}\right) \sqrt{\alpha K_\phi^\eta},$$

and

$$\begin{aligned} &\left| \int_{b'}^{b'+\eta} \sqrt{(1-\alpha)\dot{\phi}(x)} f(\phi_{\alpha,\eta}(x), x) - \sqrt{\dot{\phi}(x)} f(\phi(x), x)dx \right| \\ &\qquad \leq 2\|f\|_\infty \int_{b'}^{b'+\eta} \sqrt{\dot{\phi}(x)}dx \leq 2\sqrt{\eta}\|f\|_\infty \sqrt{K_\phi^\eta}, \end{aligned}$$

so that

$$U_f(\phi_{\alpha,\eta}) - U_f(\phi) \geq \sqrt{K_\phi^\eta} \left(\frac{2c\delta}{\sqrt{b'-a'}}\sqrt{\alpha} - 2\sqrt{\eta}\|f\|_\infty\right).$$

Hence, choosing η sufficiently small, say $\eta < \frac{4\eta^2\delta^2\alpha}{(b'-a')\|f\|_\infty}$, we get $U_f(\phi_{\alpha,\eta}) - U_f(\phi) > 0$, which is a contradiction with the definition of ϕ as a maximizer. ∎

LEMMA 5.19. *For any $a \in [0,1]$, $\nu_\phi(\{a\}) = 0$.*

Proof. Let $\phi \in \mathcal{D}_+^*$ such that $U_f(\phi) = \max_{\mathcal{D}^*} U_f$. Proposition 5.7 implies that $U_{\tilde{f}}(\phi^-) = \max_{\mathcal{D}^*} U_{\tilde{f}}$. But if f satisfies conditions [H1] and [H2], so does $\tilde{f}$, and thus, one has $\phi^- \in \mathcal{D}_+^*$. Lemma 5.8 now implies that $\phi = (\phi^-)^-$ is continuous, which concludes the proof. ∎

6. Proof of the regularity results.

Proof of Theorem 3.3. The idea of the proof is to use the fact that after a proper change of variable and rescaling, ϕ^* is the solution of a local variational problem around any point x. Hence, the behavior of ϕ^* at x depends only on the properties of the locally optimal solutions involving the values of f in a small neighborhood of $(\phi^*(x), x)$.

Let $0 \leq a < b \leq 1$, and define for any $\phi \in \mathrm{Hom}^+$ the new "focusing" functions $\phi_{a,b} \in \mathrm{Hom}^+$ and $f_{a,b}^\phi : [0,1]^2 \to \mathbb{R}$ by

$$\phi_{a,b}(x') = \frac{\phi(a + x'(b-a)) - \phi(a)}{\phi(b) - \phi(a)} \ \forall x' \in [0,1],$$

and

$$f_{a,b}^{\phi}(y',x') = f(\phi(a)+y'(\phi(b)-\phi(a)), a+x'(b-a)).$$

One has, after a simple computation,

$$\int_a^b \sqrt{\dot\phi(x)}f(\phi(x),x)dx = \sqrt{(b-a)(\phi(b)-\phi(a))}U_{f_{a,b}^{\phi}}(\phi_{a,b})$$

so that

$$U_{f_{a,b}^{\phi^*}}(\phi_{a,b}^*) = \max_{\phi\in\mathrm{Hom}^+} U_{f_{a,b}^{\phi^*}}(\phi).$$

Let $\delta_{a,b} = \|f_{a,b}^{\phi^*}\|_\infty - \int_0^1 f_{a,b}^{\phi^*}(x',x')dx'$ and assume that $\|f_{a,b}^{\phi^*}\|_\infty > 0$; then we deduce from Theorem 3.1 that

$$\|\phi_{a,b}^* - \mathrm{Id}\|_\infty \le \sqrt{1-\left(\frac{\|f_{a,b}^{\phi^*}\|_\infty - \delta_{a,b}}{\|f_{a,b}^{\phi^*}\|_\infty}\right)^2} \le \sqrt{2\frac{\delta_{a,b}}{\|f_{a,b}^{\phi^*}\|_\infty}}. \tag{6.1}$$

Hence, if $x_0\in[a,b]$ and f is Hölder continuous with parameter $\alpha>0$ at $(\phi^*(x_0),x_0)$ and if $f(\phi^*(x_0),x_0)>0$, then we get easily that there exists a constant k_{x_0} (depending only on $(\phi^*(x_0),x_0)$) such that

$$\delta_{a,b} \le k_{x_0}\max(|\phi^*(b)-\phi^*(a)|^\alpha,(b-a)^\alpha)\,.$$

Using the fact that $\|f_{a,b}^{\phi^*}\|_\infty \ge f(\phi^*(x_0),x_0)>0$, we deduce from inequality (6.1) that

$$\|\phi_{a,b}^* - \mathrm{Id}\|_\infty \le C_{x_0}(b-a)^{\alpha/2}\max\left(\left(\frac{|\phi^*(b)-\phi^*(a)|}{b-a}\right)^{\alpha/2},1\right), \tag{6.2}$$

with $C_{x_0} = \sqrt{\frac{2k_{x_0}}{f(\phi^*(x_0),x_0)}}$. Choosing $a=x_0$ and $b=x_0+h$ with any $h>0$ such that $x_0+h\le 1$, we deduce from the previous inequality that, for all $u\in]0,1]$,

$$|\Delta\phi^*(x_0,uh)-\Delta\phi^*(x_0,h)| \le \Delta\phi^*(x_0,h)C_{x_0}\frac{h^{\alpha/2}}{u}\max(\Delta\phi^*(x_0,h)^{\alpha/2},1), \tag{6.3}$$

where for any $h'>0$ we have $\Delta\phi^*(x_0,h') = (\phi^*(x_0+h')-\phi^*(x_0))/h'$. The fact that

$$\lim_{h\to0,h'>0}\Delta\phi^*(x_0,h')$$

exists and is positive is a consequence of the following lemma, applied to $F(h) = \Delta\phi^*(x_0,h)$. (Note that $hF(h)\to 0$ if $h\to0$, since ϕ^* is continuous.)

LEMMA 6.1. *Let $F>0$ be a function defined on $]0,\beta]$ (for some $\beta>0$) and such that, for all $h\in]0,\beta]$ and for all $u\in]0,1]$, and for some constants $K>0$, $\rho>0$, and $\beta>0$,*

$$|F(uh)-F(h)| \le K.F(h)(1+F(h)^\rho)\frac{h^\rho}{u}\,. \tag{6.4}$$

Assume, moreover, that $\lim_{h\to0}hF(h)=0$. Then, $\lim_{h\to0}F(h)$ exists and is strictly positive.

Proof of Lemma 6.1. Let $h_0 \in]0, \beta]$, and let, for $n \geq 1$, $v_n = F(h_0 2^{-n})$. From (6.4), we get, for some constant K',

$$|v_{n+1} - v_n| \leq K' h_0^\rho v_n (1 + v_n^\rho) 2^{-\rho n}\,. \tag{6.5}$$

Clearly, to prove that v_n converges, it suffices to prove that it is bounded. In fact, it merely suffices to prove that $v_n \leq C.2^{\gamma n}$ for some $\gamma < 1$, since, in this case, (6.5) yields an inequality of the kind

$$|v_{n+1} - v_n| \leq K'' v_n 2^{-\rho' n}$$

for some constants K'' and $\rho' > 0$, which implies in turn that v_n is bounded, since $\prod_{k=0}^{\infty}(1 + K'' 2^{-\rho' k}) < \infty$.

So, fix $\gamma < 1$ and let us prove that, if h_0 is taken to be small enough, one has, for all n, $v_n \leq F(h_0) 2^{\gamma n}$. Assuming that this is true for $n \geq 0$ (recall that $v_0 = F(h_0)$, so that it is true for $n = 0$), we show that this is true for $n + 1$. We have

$$v_{n+1} \leq F(h_0).2^{\gamma n}(1 + K' h_0^\rho (1 + F(h_0)^\rho 2^{\rho\gamma n}) 2^{-\rho n}) \leq F(h_0) 2^{\gamma n}(1 + K' h_0^\rho (1 + F(h_0)^\rho))$$

so that it suffices to take h_0 such that $1 + K' h_0^\rho (1 + F(h_0)^\rho) < 2^\gamma$ to get the desired conclusion.

Thus, v_n converges to a limit v. But since (6.5) implies that

$$v_{n+1} \geq v_n (1 - K' h_0^\rho 2^{-\rho n}),$$

we have, letting n_0 such that $K' h_0^\rho 2^{-\rho n_0} < 1$, for all $n \geq n_0$,

$$0 < v_{n_0} \prod_{k=n_0}^{\infty} (1 - K' h_0^\rho 2^{-\rho k}) \leq v_n, \tag{6.6}$$

which imples that $v > 0$.

Now, if h_n is any sequence which tends to 0 from above, one can find, for all n, an integer k_n such that $h_0 2^{-k_n - 1} < h_n \leq h_0 2^{-k_n}$, and (6.4) implies that

$$|F(h_n) - v_{k_n}| \leq K' |v_{k_n}| 2^{-\rho k_n}$$

so that, since k_n tends to infinity, $F(h_n)$ tends to v, which proves Lemma 6.1. □

We thus have proved that ϕ^* has a strictly positive right derivative denoted $\dot\phi_r^*(x_0)$. In the same way, we can prove that the left derivative denoted $\dot\phi_l^*(x_0)$ exists and is stricly positive. It thus remains to prove that both derivatives coincide. In fact, relation (6.2) with $a = x_0 - h$ and $b = x_0 + h$ yields

$$\lim_{h \to 0} \left| \frac{\phi^*(x_0) - \phi^*(x_0 - h)}{\phi^*(x_0 + h) - \phi^*(x_0 - h)} - \frac{1}{2} \right| = 0\,.$$

Since the left-hand part of the inequality also tends to $\frac{\dot\phi_l^*}{\dot\phi_l^* + \dot\phi_r^*} - \frac{1}{2}$, we get the result. Hence the first part of the theorem is proved.

Now, if f is locally uniformly Hölder continuous at x_0, there exists (since ϕ^* is continuous) an $\epsilon > 0$ in $[0, 1]$ such that (3.1) holds at point $(\phi(x), x)$ for all x such that $|x - x_0| < \epsilon$. As a consequence, ϕ^* will thus be differentiable at all such $x \in]x_0 - \epsilon, x_0 + \epsilon[$ and the increments $\Delta\phi^*(x, h)$ will converge, as $h \to 0$, uniformly to $\dot\phi^*(x)$. Since these increments are continuous, $\dot\phi^*$ is also continuous on $]x_0-\epsilon, x_0+\epsilon[$. ☐

Proof of Theorem 3.4. By Theorem 3.3, ϕ is continuously differentiable. Moreover, one has, for any ψ smooth diffeomorphism of $[0,1]$,

$$\int_0^1 \sqrt{\dot\phi}(x)\sqrt{\dot\psi}(x)f(\phi(x),\psi(x))dx \le \int_0^1 \sqrt{\dot\phi}(x)f(\phi(x),x)dx \tag{6.7}$$

(simply using that the left-hand term is $U_f(\phi\circ\psi^{-1})$). If h is any smooth function in $[0,1]$ such that $h(0)=h(1)=0$, there exists a small enough t such that $\psi(x)=x+th(x)$ is a diffeomorphism, and, after computation of the first variation in the left-hand term of (6.7), one gets that, for all smooth h with $h(0)=h(1)=0$,

$$\int_0^1 \sqrt{\dot\phi}(x)\dot h(x)f(\phi(x),x) = -2\int_0^1 \sqrt{\dot\phi}(x)\frac{\partial f}{\partial y}(\phi(x),x).h(x)dx.$$

So, letting $q(x)=\sqrt{\dot\phi}(x)f(\phi(x),x)$, one has

$$q(x)=q(0)+2\int_0^x \sqrt{\dot\phi}(u)\frac{\partial f}{\partial y}(\phi(u),u)du$$

so that q is differentiable in the ordinary sense, with derivative $2\sqrt{\dot\phi}\frac{\partial f}{\partial y}(\phi(.),.)$, which is continuous. Since

$$\dot\phi(x)=\frac{q^2(x)}{f(\phi(x),x)},$$

the numerator being positive and continuously differentiable, we get the fact that $\dot\phi$ is continuously differentiable. ☐

7. Auxiliary results.

We conclude this paper with two simple results which have important practical applications. The first one validates the possibility of implementing a matching combined with the fitting of some registration parameters. This enables us to recover some invariance properties which have not directly been incorporated in F.

The second result provides an approximation scheme, which permits us to work safely with discretized versions of a signal. It also naturally yields consistent multiscale minimization procedures, which is important for efficiency of numerical implementations.

7.1. Handling additional parameters.

In many practical situations, a matching is searched for up to some given finite-dimensional parameter which performs some registration between the two quantities which are compared. For example, in the formulation $f(\phi(x),x)=F(\theta\circ\phi(x),\theta'(x))$, one may consider that the functions θ should be identified to $\theta+b$ for any $b\in\mathbb{R}$ (in order to get a translation invariant matching), so that the complete problem becomes maximizing

$$\int_0^1 \sqrt{\dot\phi}F(\theta\circ\phi(x)+b,\theta'(x))dx$$

over all ϕ and b. For example, in [9], translation on θ represented rotations of plane curves, rotation-invariant comparison being a desirable feature for shape comparison.

More generally, we shall deal in this section with a function f, which depends on an additional extraneous parameter $\lambda \in \mathbb{R}^d$, and we shall try to find ϕ^* and λ^* which maximize

$$V_f(\phi, \lambda) = \int_0^1 \sqrt{\dot{\phi}}(x) f(\phi(x), x, \lambda) dx$$

for $\phi \in \mathcal{D}$ and $\lambda \in K$, where K is a compact subset of $\mathbb{R}^2$. Sufficient conditions for existence are provided in the following theorem. We let f_λ be the function $(x, y) \mapsto f(x, y, \lambda)$.

THEOREM 7.1. *We assume that f is continuous in $\lambda \in K$, uniformly in (x, y), and that, for all λ, the function f_λ satisfies conditions* [H1] *and* [H2] *of Theorem* 3.1. *Then, there exist $\lambda^* \in K$ and $\phi^* \in Hom^+$ such that*

$$V_f(\phi^*, \lambda^*) = \max\{V_f(\phi, \lambda) \,|\, \phi \in Hom^+, \lambda \in K\}.$$

Proof. By Theorem 3.1, for all $\lambda \in K$, the functional $\phi \mapsto V_f(\phi, \lambda)$ is upper-semicontinuous in $\phi \in \mathcal{D}^*$. Moreover, it is uniformly continuous in λ, since

$$\begin{aligned} |V_f(\phi, \lambda) - V_f(\phi, \lambda')| &\leq \int_0^1 \sqrt{\dot{\phi}} |f(\phi(x), x, \lambda) - f(\phi(x), x, \lambda')| dx \\ &\leq \sup_{x,y} |f(x, y, \lambda) - f(x, y, \lambda')|, \end{aligned}$$

which tends to 0 if λ tends to λ'. This implies that U is upper-semicontinuous as a function of the two variables ϕ and λ, and thus that there exists a maximizer $(\phi^*, \lambda^*) \in \mathcal{D}^* \times K$. Now, since ϕ^* is a maximizer of $U_f(., \lambda^*)$ over $\mathcal{D}^*$, Proposition 5.2 implies that $\phi^* \in \mathrm{Hom}^+$. □

7.2. Approximation schemes.

THEOREM 7.2. *Let $(f_n, n \geq 0)$ and f be functions defined on $[0, 1]^2 \times K$ such that*

$$\lim_{n \to \infty} \sup_{x,y,\lambda} |f_n(x, y, \lambda) - f(x, y, \lambda)| = 0.$$

Assume that all f_n and f satisfy the conditions of Theorem 7.1. *Let (ϕ_n^*, λ_n^*) be maximizers of V_{f_n} over $\mathcal{D}^* \times K$; then, there exists a subsequence of (ϕ_n^*, λ_n^*) which converges in $\mathcal{D}^* \times K$ to a maximizer (ϕ^*, λ^*) of V_f.*

Proof. Indeed, the hypotheses trivially imply that U_n converges to U uniformly on $\mathcal{D}^* \times K$, so that $\max U_n \to \max U$ and

$$\lim_{n \to \infty} U(\phi_n^*, \lambda_n^*) = \max U\,.$$

Now from (ϕ_n^*, λ_n^*) one can extract a converging subsequence in the compact space $\mathcal{D}^* \times K$ to a limit denoted (ϕ, λ), and one must have $U(\phi, \lambda) = \max U$ because U is upper-semicontinuous. □

One application of the theorem is the following. Assume, for example, that f is continuous and f_λ satisfies condition [H2] of Theorem 3.1 for all $\lambda \in K$. Let $g_n(x, y, \lambda)$ be a piecewise constant approximation of f, and let $f_n = \epsilon_n + g_n$, where ϵ_n is a sequence which tends to 0. Then, each f_n satisfies the conditions of Theorem 7.1 so that Theorem 7.2 applies. Such a situation is typical in numerical procedures.

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