A Generalized Optical Flow Constraint and its Physical Interpretation

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Abstract

This paper addresses the issue of motion estimation on image sequences. The standard motion equation used to compute the apparent motion of image irradiance patterns is an invariance brightness based hypothesis called the optical flow constraint. Other equations can be used, in particular the extended optical flow constraint, which is a variant of the optical flow constraint, inspired by the fluid mechanic mass conservation principle. In this paper, we propose a physical interpretation of this extended optical flow equation and a new model unifying the optical flow and the extended optical flow constraints. We present results obtained for synthetic and meteorological images.

1. Introduction

Nowadays, most motion estimation methods compute *optical flow*, i.e. the apparent motion, between one or several frames of an image sequence. These approaches are usually based on a motion equation called the Optical Flow Constraint (**OFC**). This equation expresses the brightness invariance over time for each point in an image sequence. This constraint has been studied by many authors, in particular [6, 8, 12]. In [10, 11], B.G. Schunck points out that the optical flow constraint is close to the mass conservation law used in fluid mechanics, then he uses it as a motion equation. This equation links density and velocity of a fluid parcel, and the volumic mass is identified as a brightness function. This motion equation is called the Extended Optical Flow Constraint (**EOFC**) which is an appropriate term as this equation is precisely the optical flow constraint with an additional divergence term. To justify this constraint, B.G. Schunck shows the surfaces observed in image sequence have a fluid behavior and the image brightness can be seen like a density. B.G. Shunck then H.H. Nagel ([9]) studied the links between the apparent motion (computed with the optical flow and the extended optical flow constraints) and the real motion of 3D-objects. More details about the relation between motion and apparent motion can be found in [4, 13]. It appears that the velocity computed from the optical flow constraint has no obvious link with real motion except in some degenerated cases. As regards the velocity computed from the extended constraint, B.G. Schunck proved that the real motion may be deduced from the optical flow when the motion is a rotation on a particular axis. However, H.H. Nagel proved in [9] that in the case of Lambertian surface, the image brightness can not be interpreted like a density and the extended optical flow constraint is not appropriate. In paper [14], Wildes *et al* compute the optical flow using the extended constraint in a specific context: working on an image sequence of fluid motion, they can measure the density from the images, so the extended optical flow constraint is identified as the mass conservation equation.

The first purpose of this paper is to propose a physical interpretation of the extended optical flow constraint: we demonstrate that such a constraint is equivalent to a *total brightness* invariance hypothesis. The total brightness is defined as the sum of grey level values of a moving object (this concept is developed in this paper). So, the extended optical flow constraint is seen as a natural extension of the optical flow constraint, this latter equation modeling the invariance of a pixel grey level value over time. The second purpose is to propose a new motion equation, unifying the optical flow constraint and the extended optical flow constraint. This equation is called the Generalized Optical Flow Constraint (**GOFC**). Such a generalization is necessary if we wish to apply the total brightness conservation principle to moving objects.

The paper is organized as follow. In section 2 we briefly present the optical flow constraint and the extended optical flow constraint. The two methods are compared. In section 3 the total brightness conservation principle is presented and we show that such a hypothesis is equivalent to the extended optical flow constraint. Section 4 is devoted to the numerical implementation of this motion equation. In section 5, we present an experimental study of the generalized optical flow constraint. Results are given for a synthetic example and for meteorological infrared data where the generalized constraint has a physical signification. Finally we provide a conclusion and give some research perspectives in section 6.

2. EOFC versus OFC

In this section, we present and briefly compare the classical approach to optical flow computation and the extended version of the optical flow constraint. However, an exhaustive comparison of these methods can be found in [3].

2.1. The Optical Flow Constraint

As stated in the introduction, this constraint expresses the fact that the grey value of each pixel is constant over time. This is modeled by the equation:

$$\frac{dI(x,y,t)}{dt} = 0$$

where I denotes the brightness function of an image sequence. The (x, y) variables are the spatial direction of the image. These coordinates are functions of t for each moving pixel. Using the chain rule derivation, we have:

$$\frac{dI(x(t),y(t),t)}{dt} = \frac{\partial I}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial I}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial I}{\partial t}$$

Which leads to the equation:

$$\nabla I.w + I_t = 0 \qquad (1)$$

where ∇ denotes the gradient operator and $w = \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}\right)$ is the velocity vector. Equation (1) is called the Optical Flow Constraint. To solve this equation, we build an energy functional that has to be minimized:

$$E_1(w) = \int_\Omega \left((\nabla I.w + I_t)^2 + \alpha||\nabla w||^2\right) dxdy \qquad (2)$$

The second term under the integration comes from the fact that equation (1) is underconstrained and we need a second assumption to solve it. The choice of the additional constraint is not simple and greatly depends on the application concerned. In general, the constraint involves the spatial variation of w with linear operators [6] or non linear operators [1, 2, 7, 8].

In functional (2) we constrain the spatial variations of w in L_2 as used by Horn *et* al. It can be demonstrated that the minimum of functional (2) is a solution of equation (1). Finding the minimum of functional (2) is equivalent to computing the zero of the Euler-Lagrange equations of this functional (i.e. the differential of E_1 w.r.t. w). A Gauss-Seidel scheme is used to compute the solution Euler-Lagrange equations (see [6] for details).

2.2. The Extended Optical Flow Constraint

The extended optical flow constraint is actually the mass conservation equation of a perfect fluid, whose relation between the density ρ and the velocity $\vec{u}$ is:

$$\frac{1}{\rho}\frac{d\rho}{dt} + \mathrm{div}(\vec{u}) = 0 \qquad (3)$$

If the image brightness I replaces the density ρ, and equation (3) is developed , we obtain the following equation:

$$\nabla I.w + I_t + I\,\mathrm{div}(w) = 0 \qquad (4)$$

This equation is very similar to equation (1) and it is called the extended optical flow constraint. Schunck ([10]) and Nagel ([9]) probably used this equation because they were interested to detecting the deformable motion (very similar to fluid behavior) but we stress Schunck do not give a physical justification for a such equation. One justification proposed by Schunck is that the real motion of a 3D-object can be estimated from the extended optical flow constraint when the motion is a rotation with the camera direction as the axis. To compute the solution to the extended optical flow constraint, we use the same approach described in the previous section (see [9, 10, 11] for details).

In subsection 5.1, we comment one some results of the optical flow and extended optical flow models.

3. Total Brightness Conservation

As proposed in the introduction, this section provides a physical signification of the constraint.

Total brightness of an object

First, we introduce the total brightness quantity. The idea is to extend the pixel grey value notion to an object. Of course, we first have to know the location of the objects in the image sequence, which is not an easy task (see section 5). Let $\mathcal{O} \subset \Omega$ be an object, Ω denotes the spatial domain of an image sequence. The total brightness of $\mathcal{O}$ is defined by the quantity:

$$\mathcal{I}_{\mathcal{O}}(t) = \int_{\mathcal{O}} I(X,Y,t)dXdY$$

where X and Y are the spatial coordinates of Ω.

The total brightness conservation hypothesis

Instead of assuming that a point has a constant brightness over time, we assume that a moving object point has locally a total brightness constant over time which is expressed by the mathematical relation:

$$\forall \mathcal{S} \subset \mathcal{O}, \ \frac{d\mathcal{I}_{\mathcal{S}}(t)}{dt} = 0 \quad (5)$$

where $\mathcal{S}$ is a (topological opened) part of $\mathcal{O}$. Such an assumption may be viewed as an extension of the pixel brightness invariance operating at the object level. It allows the objects to change their brightness while, consequently, changing their size. The pixel brightness invariance and the total brightness invariance principles are not equivalent. However, a rigid object with a constant brightness also verifies the total brightness invariance principle.

Link with the Extended Optical Flow Constraint

Let us introduce the function φ describing the position at time $t > 0$ of a point located at (x, y) coordinate at time $t = 0$. In other words, φ represents the trajectories of all points over the image sequence. φ is defined on $I\!R^2 \times I\!R^+$ and takes its values on $I\!R^2$. Let $(X, Y) = \varphi(x, y, t)$. The total brightness of a part $\mathcal{S}$ of an object becomes:

$$\mathcal{I}_{\mathcal{S}}(t) = \int_{\mathcal{S}} I \circ \varphi(x, y, t) J_{\varphi}(x, y, t) dxdy$$

with J the Jacobian operator. Equation (5) is then equivalent to:

$$\forall \mathcal{S} \subset \mathcal{O}, \quad \int_{\mathcal{S}} \frac{d}{dt}(I \circ \varphi(x, y, t) J_{\varphi}(x, y, t)) dxdy = 0$$

We apply the chain rule derivation and obtain:

$$\forall \mathcal{S} \subset \mathcal{O},$$
$$\int_{\mathcal{S}} \left((\nabla I \circ \varphi . \varphi_t + I_t \circ \varphi) J_{\varphi} + I \circ \varphi \frac{dJ_{\varphi}}{dt} \right) dxdy = 0$$

We use the approximation $\frac{dJ_{\varphi}}{dt} \sim \operatorname{div}\left(\frac{d\varphi}{dt}\right) J_{\varphi}$ which is true for a small value of J_{φ} (which can be interpreted as a small variation of φ: hence, small velocity values). Equation (5) is equivalent to:

$$\forall \mathcal{S} \subset \mathcal{O},$$
$$\int_{\mathcal{S}} (\nabla I \circ \varphi . \varphi_t + I_t \circ \varphi + I \circ \varphi \operatorname{div}(\varphi_t)) J_{\varphi} dxdy = 0$$

and the total brightness conservation ($\forall \mathcal{S} \subset \mathcal{O}, \ \mathcal{I}'_{\mathcal{S}} = 0$) can be rewritten:

$$\forall (x, y) \in \mathcal{O}, \quad \nabla I \circ \varphi . \varphi_t + I_t \circ \varphi + I \circ \varphi \operatorname{div}(\varphi_t) = 0 \quad (6)$$

As $\varphi_t = w$, where w is the velocity field, this equation becomes:

$$\forall (x, y) \in \mathcal{O}, \quad \nabla I . w + I_t + I \operatorname{div}(w) = 0 \quad (7)$$

This equation is exactly the extended optical flow constraint (4) except that it is only valid on objects $\mathcal{O}$ and not for the complete image domain Ω as with equation 4.

Equation (7) is not invariant if a constant value is added to brightness function I. So it is more rigorous to assume the conservation of total intensity of $I - I_{min}$ where I_{min} is a constant level reference value. I_{min} describes the brightness of an object having a constant total brightness and of infinite extension. Although I_{min} is not necessary equal to 0, we make this assumption from now on.

4. Global framework to compute the optical flow

The motion equation deduced from the total brightness principle is quite different from the extended optical flow constraint because it is only valid for moving objects with a constant total brightness value. Our objective is to use a generalized model featuring simultaneously the standard optical flow constraint and the extended one.

The motion equation

The pixel brightness invariance equation and the total brightness equation are very similar. As the total brightness invariance is applied only on objects, the pixel brightness invariance should be applied on the other parts of the image. This leads to the new constraint:

$$\forall (x, y) \in \mathcal{O}, \quad \nabla I . w + I_t + I \operatorname{div}(w) 1\!\!1_{\mathcal{O}} = 0 \quad (8)$$

where $1\!\!1_{\mathcal{O}}(x, y) = 1 \Longleftrightarrow (x, y) \in \mathcal{O}$ and 0 else. We call this motion equation the Generalized Optical Flow Equation, because it unifies the two different optical flow constraints.

Numerical resolution

Equation (8) is solved using a variational approach: the solution is determined as the minimum of a functional. Since equation (8) is underconstrained, a second term is added to the functional. The second term takes into account the L_2 norm of the velocities' spatial variation:

$$E_2(w) = \int_{\Omega} (\nabla I . w + I_t + I \operatorname{div}(w) 1\!\!1_{\mathcal{O}})^2 \, dxdy + \beta \int_{\Omega} ||\nabla w||^2 dxdy$$

where $||.||$ denotes the Euclidian norm of a vector, i.e. $||\nabla w||^2 = u_x^2 + u_y^2 + v_x^2 + v_y^2$ and β is a parameter weighting the effect of the second term which can be seen as a regularizing term. This parameter is initialized empirically. The minimum value of E_2 provides a solution to equation (8). The Euler-Lagrange equations associated to E_2 are:

$$\left\{ \begin{array}{l} -\frac{\partial}{\partial x}\left(I\left(\nabla I.w+I_t+I\operatorname{div}(w)\mathbb{1}_{\mathcal{O}}\right)\right)-\beta\triangle u=0 \\ -\frac{\partial}{\partial y}\left(I\left(\nabla I.w+I_t+I\operatorname{div}(w)\mathbb{1}_{\mathcal{O}}\right)\right)-\beta\triangle v=0 \end{array} \right. \tag{9}$$

where $\triangle$ denotes the Laplacian operator. The Green formula is required to determine the Euler-Lagrange equations, we suppose the brightness function is null on image boundaries so the the boundary term of the Green formula vanishes. To solve system (9), we use the following scheme:

$$\left\{ \begin{array}{ll} u^{k+1}= & u^k-\frac{\partial}{\partial x}\left(I\left(\nabla I.w^k+I_t+I\operatorname{div}(w^k)\mathbb{1}_{\mathcal{O}}\right)\right) \\ & -\beta\triangle u^k \\ v^{k+1}= & v^k-\frac{\partial}{\partial y}\left(I\left(\nabla I.w^k+I_t+I\operatorname{div}(w^k)\mathbb{1}_{\mathcal{O}}\right)\right) \\ & -\beta\triangle v^k \end{array} \right. \tag{10}$$

A stationary solution of system (10) is a solution of equation (9). In fact, we do not use this approach to compute w because it leads to a complex numerical scheme. We suppose the case where $\mathbb{1}_{\mathcal{O}}=1$. Equation (8) is equivalent to:

$$\operatorname{div}(wI)+I_t=0 \tag{11}$$

if we suppose $I \not\equiv 0$. We make the change of variable $W=Iw$ and we minimize the functional:

$$E_3(W)=\int_{\mathcal{O}}\left[\left(\operatorname{div}(W)+I_t\right)^2+\beta\|W\|^2\right]dxdy \tag{12}$$

As the brightness function I does not appear inside the differential part of the equation (11), the Euler-Lagrange equation associated to (11) is now very simple:

$$\left\{ \begin{array}{l} -\frac{\partial}{\partial x}(I_t+\operatorname{div}(W))-\beta\triangle U=0 \\ -\frac{\partial}{\partial y}(I_t+\operatorname{div}(W))-\beta\triangle V=0 \end{array} \right. \tag{13}$$

The solution of the previous equation system is given by the following numerical scheme computed for each point inside $\mathcal{O}$:

$$\left\{ \begin{array}{l} U^{k+1}=U^k-\frac{\partial}{\partial x}\left(I_t+U_x^k+V_y^k\right)-\beta\triangle U^k \\ V^{k+1}=V^k-\frac{\partial}{\partial y}\left(I_t+U_x^k+V_y^k\right)-\beta\triangle V^k \end{array} \right. \tag{14}$$

Motion is computed with two numerical schemes: the scheme (14) for points inside $\mathcal{O}$ and the scheme of brightness invariance (see [6]) for points outside $\mathcal{O}$.

The choice of the segmentation process

Using the generalized motion model, the choice of the segmentation process, i.e. the location of objects, is very important and depends on the context. We remark that the bigger the contribution of the divergence term is, the higher the value of I is. So, for a small value of I, the extended optical flow constraint becomes very similar to the optical flow constraint. A basic idea, in order to compute $\mathbb{1}_{\mathcal{O}}$, is to apply a threshold of the lowest grey level values.

5. Experimental results

5.1. Synthetic examples

The pixel brightness invariance and the total brightness invariance are not equivalent. In particular, the pixel brightness invariance does not imply the total brightness invariance and inversely. We built a synthetic sequence verifying the total brightness invariance : this is a growing square with grey level values decreasing over time in order to keep a total brightness constant (see figure 1) and having a small translational motion (from left to right). Figure 2 depicts the result of computation of the brightness invariance model (optical flow constraint) between images 1 and 2 of the sequence and the result using the total brightness invariance model (extended optical flow). In this case, the total brightness model is equivalent to the generalized model because the grey level value of a background pixel is equal to zero, so the term $I\operatorname{div}(w)$ vanishes. The total brightness invariance model gives better results because it captures correctly the model on the left side of the square and inside the square. Moreover, the velocity field of the extended optical flow is smoother than the optical flow because the divergence term is an additional spatial regularizing constraint adapted to this case.

5.2. The meteorological example

In this section, we experiment the generalized optical flow constraint on remote sensing data. These data were acquired by the European METEOSAT satellite in an infrared channel. In these images, a grey level value is a measure of temperature. Regarding the clouds observed on infrared data, it has been proved that the cloud temperature is directly linked to the elevation leading a measure of cloud elevation. The total brightness invariance principle is equivalent to a volume conservation (see figure 3). Quantity $\mathcal{I}_{\mathcal{O}}$ is an approximation of the volume of $\mathcal{O}$ as we only have access to the top of clouds. It could be interesting to suppose a local volume conservation of clouds on infrared data because such an assumption is similar to the mass conservation (at mesoscale, we have the same equation (3) between the volumic mass and the volume).

Figures 4, 5 and 6 illustrate respectively the result of the pixel brightness invariance, the total brightness invariance and the generalized models. To compute the generalized model, we used a Markovian segmentation model based on brightness homogeneity property [5] to locate efficiently the cloudy structures within images.

Comparing the first two results, the total brightness invariance hypothesis gives interesting results from a meteorological point of view. First, it detects the local deformation of the cloud, especially on the boundary, while the classical optical flow model cannot. Second, singular points are detected using this model. These points are the centers of local expansion or contraction of the clouds and they can be detected throughout the image sequence. The total brightness invariance and the generalized models give similar results but there are some differences:

- only the motion captured in the background is different due to the motion model used,
- on some cloud areas, the generalized model gives better result than the total brightness model because it is computed only on cloud area: the background area does not infer on the velocity estimation (dark pixels attract white pixels).

Another problem concerns the estimation of velocity norm. All methods presented here (optical flow, extended optical flow, generalized optical flow constraints) under-evaluate the velocity norm on meteorological data. This problem is mainly due to the fact that L_2 regularizing constraint is not really adapted to these data. Another adapted way to constrain w should be investigated.

6. Conclusion and perspectives

In this paper, we proposed an original interpretation of the Extended Optical Flow Constraint. This constraint is equivalent to a total brightness conservation principle (with some conditions) and this principle extends the pixel brightness conservation principle modeled by the Optical Flow Constraint. We have seen that the total brightness conservation principle implies knowledge of the location of the moving objects, or in other words, the areas where the extended optical flow constraint is valid and the areas where the optical flow constraint is valid. In general, a motion detection process should be used to determinate the active temporal regions. In the meteorological framework, presented as an experimental application, a simple threshold or a complex cloud segmentation (to enhance the results) process can be used. Thus, the total brightness conservation principle leads to a new motion equation: the Generalized Optical Flow Constraint. This equation can be seen like as a unified model between the **EOFC** and the **OFC** models:

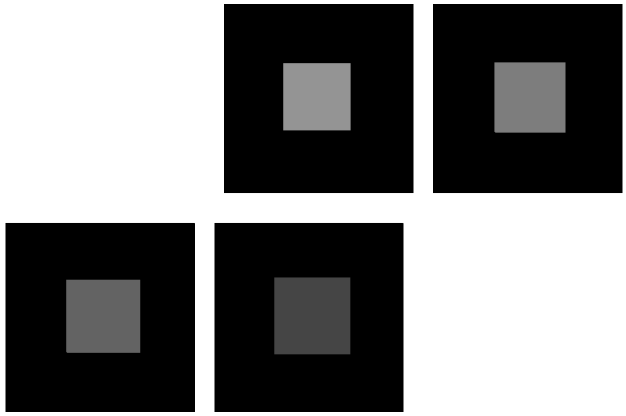

Figure 1. Sequence of a growing square with a constant total brightness and a translational motion.

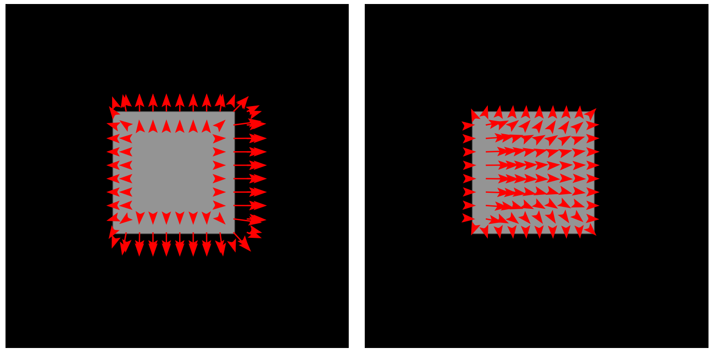

Figure 2. Left : result of the pixel brightness invariance model between frames 1 and 2 of the square sequence ($\alpha = 100$). Right : result of total brightness invariance model ($\beta = 40$).

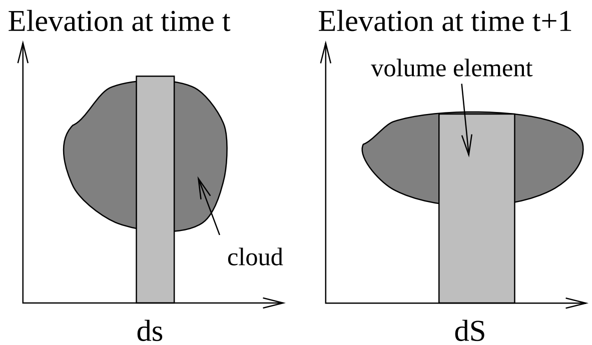

Figure 3. Conservation of volume on meteorological infrared data.

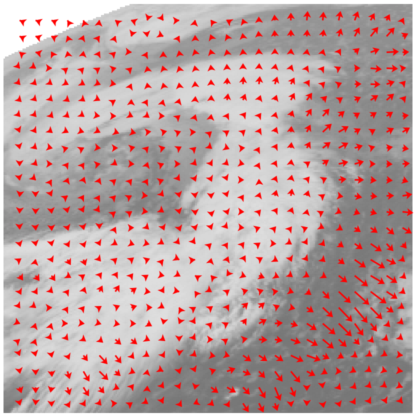

Figure 4. Result of the pixel brightness invariance model on an infrared image ($\alpha = 40$).

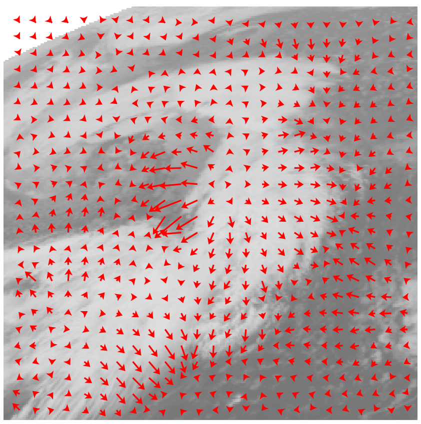

Figure 5. Result of the total brightness invariance model ($\alpha = 100, \beta = 40$).

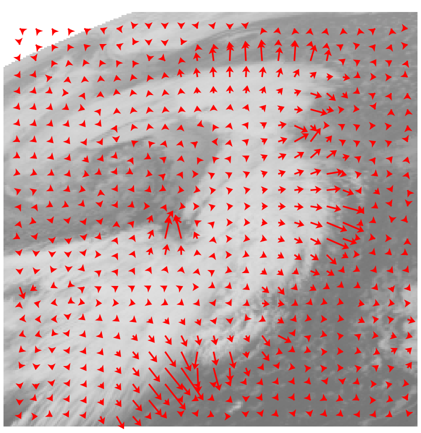

Figure 6. Result of the generalized model.

the optical flow constraint is applied on pixels verifying a brightness invariance and the extended optical flow constant is applied on areas verifying a total brightness invariance. As for the total brightness invariance model, we need a segmentation process to locate areas where the total brightness invariance principle is valid. The results obtained with the **GOFC** seen accurate on meteorological data. To overcome the problem of the bad estimation of the velocity norm on meteorological infrared data, we have developed an implementation of the generalized optical flow model using an affine priori motion model instead of constraining the spatial variation of w. This approach provides a correct velocity norm estimation.

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