Image and Vision Computing 17 (1999) 381–389

Optimal matching between shapes via elastic deformations

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Received 26 June 1997; received in revised form 5 March 1998; accepted 5 May 1998

Abstract

We describe an elastic matching procedure between plane curves based on computing a minimal deformation cost between the curves. The design of the deformation cost is based on a geodesic distance defined on an infinite dimensional group acting on the curves. The geodesic paths also provide an optimal deformation process, which allows interpolation between any plane curves. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Time warping; Elastic matching; Infinite dimensional group action

1. Introduction

Deformable shapes arise naturally in image analysis, especially for applications in medical or biological imaging. To detect, recognize, or simply organize such shapes (for example in a database), there is an obvious need for comparison tools. In this paper, we address the problem of comparing and matching silhouettes (i.e. plane curves) with the help of a variational approach on an infinite dimensional deformation group acting on them.

When comparing non-rigid objects, there cannot be any natural notion of intrinsic parametrization (or labelling) of the outline, as in Ref. [1], and most of the refined shape comparison methods go through a step in which a common labelling is obtained by matching the different structures within the shapes. We shall represent a shape outline as a parametric plane curve, formally expressed as a function

$$(m : s \in I \rightarrow m(s) \in \mathbb{R}^2)$$

I being a bounded interval. Given another curve $\tilde{m}$ defined on another interval $\tilde{I}$, the matching problem is thus to find a one-to-one correspondence $u \leftrightarrow \tilde{u}$ between the abscissa of the points of m and the points of $\tilde{m}$. In other terms, one must determine a one-to-one mapping ϕ from I onto $\tilde{I}$.

In most applications, the approach to estimate ϕ is variational. Assuming enough regularity (for example, that it is a diffeomorphism), one performs the minimization of some functional $E(\phi,m,\tilde{m})$ which generally creates a trade-off between the similarity of the functions m and $\tilde{m} \circ \phi$ and some measure of the regularity of ϕ. For example, denoting by κ, and $\tilde{\kappa}$ the curvature functions of m and $\tilde{m}$, the functional used in Ref. [2] is

$$E = \frac{1}{2}\int (\tilde{\kappa} \circ \phi(u) - \kappa(u))^2 du + \int \| \frac{\partial}{\partial u}(\tilde{m} \circ \phi(u) - m(u))\|^2 du$$

Very often, the regularity constraint is on the derivative of ϕ, and we shall speak (with a loose analogy) of ''elastic matching'' (see also Ref. [3]).

On the basis of a group theoretic-based approach to the problem, the technique we propose will also lead to a matching through the minimization of a cost function. The deformation of a plane curve is modelled as the action of an infinite dimensional group which involves the diffeomorphism ϕ. The acting group being equipped with a suitable Riemannian metric, we apply a least action principle, that is, we look for the element in the group which is closest to the identity among all those which transform one curve into another, and let the cost function be the associated distance. It is easy to provide sufficient conditions under which the resulting distance is symmetrical and satisfies the triangular inequality (see the next section). However, such a framework, which has essentially been introduced for computer vision applications by Ulf Grenander and his collaborators (cf. for example, Ref. [4]), generally leads to intractable computations, and one of the interests of our contribution is to exhibit naturally arising deformation groups and metrics which yield a closed-form formula for the cost function.

Since the matching procedure is global, the whole outlines must be known in order to perform the comparison. As

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PII: S0262-8856(98)00125-5

a consequence, the method cannot be directly used for recognizing occluded objects.

The construction of the cost function has been done in Ref. [5], in which the motivation was to define a true distance between plane curves. We focus here on the performance of the cost function for elastic matching, and its ability to provide a ‘‘minimal cost’’ interpolation between two shapes. We start by describing a general algebraic and variational framework related to the construction of the distance. We then concentrate on a case which leads to closed-form expressions.

2. Distance and transitive group action

2.1. Definition

A distance over a set M is a mapping $d{:}M^2 \mapsto \mathbb{R}_+$ such that, for all $m,m',m'' \in \mathcal{C}$

(D1) $d(m,m') = 0 \Leftrightarrow m = m'$

(D2) $d(m,m') = d(m',m)$

(D3) $d(m,m'') \le d(m,m') + d(m',m'')$.

If (D1) is reduced into the property that $d(m,m) = 0$ for all m, we say that d is a *pseudo-distance*.

If G is a group acting on M, the distance d is G-invariant if and only if, for all $g \in G$, for all m, $m' \in M$, we have $d(g.m,g.m') = d(m,m')$.

2.2. Algebraic formulation

In this abstract setting, assume that the deformation of a curve m may be represented through the action of a group G on m, which associates to an element g in G and to a curve m, a deformed curve $g.m$. Let e be the identity element in G.

Denote by $\mathcal{C}$ the space containing the curves which are taken into consideration, so that G acts on $\mathcal{C}$. We shall assume that it is possible to deform any object into any other one, that is, that the group action is transitive: for all m and $\tilde{m}$, there exists $g \in G$ such that $\tilde{m} = gm$.

In this framework, the set $\mathcal{C}$ may be identified to the quotient space G/G_m, where G_m is the isotropy subgroup of a reference element $m \in \mathcal{C}$:

$$G_m = \{g \in G, g.m = m\}.$$

We recall that if H is a subgroup of G, the quotient space G/H is the set of cosets:

$$G/H = \{g.H, g \in G\}$$

The group G acts naturally on G/H by letting $\tilde{g}.[g.H] = (\tilde{g}g).H$. From elementary group theory, we have:

Proposition 1. *Let G be a group acting transitively on a set $\mathcal{C}$. Let $m \in \mathcal{C}$ and G_m be the isotropy subgroup of m. The mapping*

$$\Psi_m : G/G_m \to \mathcal{C}$$

$$g.G_m \mapsto g.m$$

is a bijection which carries the action of G on G/G_m to the action of G on $\mathcal{C}$.

The last sentence means that $\Psi_m(\tilde{g}.(g.G_m)) = \tilde{g}.\Psi_m(G_m)$.

With this proposition, we are able to link (pseudo-)distances on $\mathcal{C}$ to (pseudo-)distances on G which are left-invariant by the isotropy subgroup of some reference element $m_0 \in \mathcal{C}$. We thus fix such an m_0 for the rest of this section. We let $H = G_{m_0}$. A (pseudo-)distance d_G on G is invariant by the *right* action of H on G if and only if, for all $g, \tilde{g} \in G$ and all $h \in H$,

$$d_G(gh, \tilde{g}h) = d_G(g, \tilde{g}).$$

We have

Proposition 2. *Let d_G be a distance on G, invariant by the right-action of $H = G_{m_0}$. The function d, defined by, for all $m,m' \in \mathcal{C}$*

$$d(m,m') = \inf\{d_G(g,\tilde{g}), g.m_0 = m, g'.m_0 = m'\} \quad (1)$$

is a pseudo-distance on $\mathcal{C}$.

Let us briefly check the triangular inequality (D3). Let $m, m', m''\mathcal{C}, g, g_1', g_2', g'' \in G$ be such that

$$m_0 = g^{-1}.m = (g_1')^{-1}.m' = (g_2')^{-1}.m' = (g'')^{-1}.m'' \quad (2)$$

We have

$$\begin{aligned} d_G(g,g'') &\le d_G(g,g_1') + d_G(g_1',g'') \\ &= d_G(g,g_1') + d_G(g_2', g''(g_1')^{-1}g_2') \end{aligned} \quad (3)$$

because $(g_1')^{-1}g_2' \in H$. Minimizing this quantity with respect to g, g_1', g_2', g'' gives $d(m,m'') \le d(m,m') + d(m',m'')$.

In Eq. (3) we see why it was necessary to switch from left-action of G on $\mathcal{C}$ to right-action of H on G. However, from the remark that, if $(g,m) \mapsto g.m$ is a left-action, then $(g,m) \mapsto g^{-1}.m$ is a right-action, we have

Proposition 3. *Let d_G be a distance on G, invariant by the left-action of $H = G_{m_0}$. The function d, defined for all m, $m' \in \mathcal{C}$.*

$$d(m,m') = \inf\{d_G(g,g'), m_0 = g.m, m_0 = g'.m'\} \quad (4)$$

is a pseudo-distance on $\mathcal{C}$.

Another point of view is to define a distance on $\mathcal{C}$ through a minimal effort criterion. In this framework, a cost is associated to each deformation $m \to g.m$, and denoted by $\Gamma(g,m)$. If m and $\tilde{m}$ are two curves, we let

$$d(m,\tilde{m}) = \inf\{\Gamma(g,m), g \in G, g.m = \tilde{m}\} \quad (5)$$

This is a distance under the conditions listed in the next proposition:

Proposition 4. *If* Γ *is such that*

(*C1*) $\Gamma \geq 0$ *and* $\Gamma(e,m) = 0$
(*C2*) $\Gamma(g,m) = \Gamma(g^{-1},gm)$
(*C3*) $\Gamma(gh,m) \leq \Gamma(h,m) + \Gamma(g,hm)$

then, d defined by Eq. (5) *is a pseudo-distance on* $\mathcal{C}$.

In fact, both approaches (through Eqs. (2) and (5)) are equivalent:

Proposition 5. *If* Γ *satisfies* (*C1*), (*C2*) *and* (*C3*), *then for all* $m_0 \in \mathcal{C}$, *the function* d_G *defined by*

$$d_G(g,\tilde{g}) = \Gamma(\tilde{g}g^{-1}, g.m_0)$$

is a distance on G, invariant by the right-action of G_{m_0}. *Conversely, starting from* d_G, *invariant by the right-action of* G_{m_0}, *one defines an effort function satisfying* (*C1*), (*C2*) *and* (*C3*) *by letting*

$$\Gamma(h,m) = d_G(hg,g)$$

in which g is any element of G such that $g.m_0 = m$.

A left-invariant version of the above goes as follows. First, one needs to set, instead of Eq. (5),

$$d(m,\tilde{m}) = \inf\{\Gamma(g,m), g \in G, g^{-1}.m = \tilde{m}\}. \tag{6}$$

Conditions (C1) and (C2) are unchanged and (C3) becomes

(C3′) $\Gamma(gh,m) \leq \Gamma(g,m) + \Gamma(h,gm)$.

And we have:

Proposition 6. *If* Γ *satisfies* (*C1*), (*C2*) *and* (*C3′*), *then for all* $m_0 \in \mathcal{C}$, *the function* d_G *defined by*

$$d_G(g,\tilde{g}) = \Gamma(\tilde{g}^{-1}g, g^{-1}.m_0)$$

is a distance on G, invariant by the left-action of G_{m_0}. *Conversely, starting from* d_G, *invariant by the left-action of* G_{m_0}, *one defines an effort function satisfying* (*C1*), (*C2*) *and* (*C3′*) *by letting*

$$\Gamma(h,m) = d_G(gh,g)$$

in which g is any element of G such that $g.m = m_0$.

This presentation generalizes the concept of ''effort functional'' introduced by Grenander in Ref. [6]. In this reference, an effort functional is defined as a cost function Γ satisfying conditions (C1) to (C3), in the particular case when $\Gamma(g,m)$ does not depend on the deformed curve m. One can see that this is equivalent to considering distances δ on G which are not only invariant under the action of the isotropy group of a reference curve m_0, but also under the left-action of the whole group G (left-invariant distances). However, in some cases, it is natural to also consider situations for which the deformation costs may also depend on the deformed object. For example, when a notion of size is present, deforming large objects may cost more than deforming small ones.

The above construction comprises one part of the analysis which is needed to construct ''least action distances'' on our set $\mathcal{C}$. The other part will be more constructive, and relies on some differential geometry concepts, leading to a variational formulation.

2.3. Infinitesimal cost

This section may be disregarded at first reading, or by the reader who is unfamiliar with elementary differential geometry (of Ref. [7],[8],[9]); in this case go directly to Section 2.4.

Assume that G is something like a Lie group and $\mathcal{C}$ something like a differentiable manifold, on which one can speak safely of ''small variations'' of elements. In particular, given a path $t \mapsto \mathbf{g}(t)$, defined on $[a,b]$, with values in G, one is able to define the derivative $\mathbf{g}'(t)$. This $\mathbf{g}'(t)$ is a tangent vector to G at the point $\mathbf{g}(t)$. Assume that, for all $g \in G$, one provides a scalar product $\langle .,. \rangle_g$ on the vector space of all tangent vectors at g to G (that is, a Riemannian metric on G). The length of the path $\mathbf{g}$ is then given by

$$L(\mathbf{g}) = \int_a^b \|\mathbf{g}'(t)\|_{\mathbf{g}(t)} dt$$

and the (Riemannian) distance on G between two elements g and $\tilde{g} \in G$ is defined as the length of the shortest path linking g and $\tilde{g}$.

For $h \in G$, we define R_h to be the right-translation $g \mapsto gh$. Similarly, L_h is the left-translation $g \mapsto hg$. We assume that these functions are differentiable. The differential of R_h at $g \in G$ will be denoted $d_g R_h$. It associates a tangent vector $d_g R_h.X$ at gh to a tangent vector X at g, and measures, in some sense, in which direction gh varies when g varies slightly in the direction X. A Riemannian metric on G is invariant by the right-action of a subgroup H of G if, for all $g \in G$, for all tangent vectors $X, Y \in G$, for all $h \in H$,

$$\langle X, Y \rangle_g = \langle d_g R_h.X, d_g R_h Y \rangle_{gh}.$$

If this is true, the associated Riemannian distance is invariant by the right-action of H. Left-invariance is handled by replacing R_h by L_h.

Given a Riemannian metric on G (invariant by the right-action of $H = G_{m_0}$), and the associated distance d_G, we have defined

$$\Gamma(\tilde{g},m) = d_G(\tilde{g}g,g)$$

where $g.m_0 = m$. This is given by the minimum of

$$\int_a^b \|\mathbf{g}'(t)\|_{\mathbf{g}(t)} dt$$

over all paths $\mathbf{g}:[a,b] \to G$ which link g to $\tilde{g}g$. Now, $\mathbf{g}$ is a path linking e to $\tilde{g}$, if and only if $\mathbf{g}.g$ is a path linking g to $\tilde{g}.g$

so that $\Gamma(\tilde{g}, m)$ can be obtained by minimizing, over all path $\mathbf{g}$ linking e and $\tilde{g}$, the quantity

$$W(\mathbf{g}, m) = \int_a^b \|d_{\mathbf{g}(t)} R_g.\mathbf{g}'(t)\|_{\mathbf{g}(t).g} dt.$$

For $m \in \mathcal{C}$, $\tilde{g} \in G$, and a tangent vector X to G at $\tilde{g}$, let

$$N_{\tilde{g}}(X, m) = \|d_{\tilde{g}} R_g.X\|_{\tilde{g}g}$$

for any g such that $g.m_0 = m$. We can write

$$W(\mathbf{g}, m) = \int_a^b N_{\mathbf{g}(t)}(\mathbf{g}'(t), m) dt.$$

But, we have

$$N_{\tilde{g}}(X, m) = N_e(d_{\tilde{g}} R_{\tilde{g}^{-1}} X, \tilde{g}.m)$$

so that

$$W(\mathbf{g}, m) = \int_a^b N_e(d_{\mathbf{g}(t)} R_{\mathbf{g}(t)^{-1}}.\mathbf{g}'(t), \mathbf{g}(t).m) dt.$$

So the cost function $\Gamma(g,m)$, which comes after minimizing $W(\mathbf{g},m)$ over all paths linking e and g, is completely specified once the quantities $N_e(X,m)$ are defined, for all $m \in \mathcal{C}$ and all tangent vector X to G at e. The interest in expressing W and Γ in this form is that $N_e(X,m)$ has a very neat intuitive interpretation: it provides the infinitesimal cost of slightly deforming m in the direction X. Most of the time, these small deformation costs are available before designing the distance, or, at least, are easy to conceive. The construction above fills the gap between such small deformation costs and large rigorous deformation costs leading to distances.

If one prefers to use only left translations, as we do, (see Section 3), one simply needs to replace the definition of W by

$$W(\mathbf{g}, m) = \int_0^1 N_e\left((d_e L_{\mathbf{g}(t)})^{-1} \frac{d\mathbf{g}}{dt}(t), \mathbf{g}(t)^{-1}.m\right).$$

2.4. Summary

Let us summarize (in the left-translation case) the steps to be followed for constructing a least-deformation distance in the above framework:

We are given the object space $\mathcal{C}$.

- Fix a group G which acts on $\mathcal{C}$ (''deformation group''), such that any object can be deformed into any other.
- Let e be the identity element in G, and use the notation $e + tX$ to symbolize a small variation of e in the direction X (t small). For each $m \in \mathcal{C}$, one must define the cost of the deformation $m \mapsto (e + tX).m$, in such a manner that it is proportional to t, and provides a norm for X. It will be denoted by $N_e(X, m)$.
- For all $g \in G$, compute the differential of the transformation $L_g : \tilde{g} \mapsto g\tilde{g}$.
- Define the cost of a path $\mathbf{g}$:[0,1] $\to G$ deforming an object m $\in \mathcal{C}$ by

$$W(\mathbf{g}, m) = \int_0^1 N_e\left((d_e L_{\mathbf{g}(t)})^{-1} \frac{d\mathbf{g}}{dt}(t), \mathbf{g}(t)^{-1}.m\right). \quad (7)$$

- Define the cost between of a deformation $m \mapsto g.m$ by

$$\Gamma(g, m) = \inf_{\mathbf{g}} \ W(\mathbf{g}, m),$$

the infimum being searched over all paths $\mathbf{g}$:[0,1] $\to G$, such that $\mathbf{g}(0) = e$ and $\mathbf{g}(1) = g$.

- Define the distance between two objects m and $\tilde{m}$ in $\mathcal{C}$ by

$$d(m, \tilde{m}) = \inf\{\Gamma(g, m), g \in G, g^{-1}.m = \tilde{m}\}.$$

2.5. Remarks

(1) This framework, through Eqs. (1) and (5), only provides pseudo-distances. The additional property, that

$$d(m, m') = 0 \Rightarrow m = m'$$

cannot be obtained in a general context. It will be true however for the distance we shall introduce in the next section.

(2) Although the previous section provides a constructive method for obtaining distances on $\mathcal{C}$, it may, however, involve intractable computations since it requires the minimization of a functional defined on the set of all paths on the acting group G. It is quite lucky that this program, when applied to plane curves, leads to a closed form formula for W, and thus to a much simpler variational problem, for the computation of d.

3. Comparing plane curves

3.1. Generalities

We consider plane curves with arc-length parametrization, i.e. differentiable functions (m:[0,L] $\to \mathbb{R}^2$), with a normed gradient ($\|\dot{m}_s\| \equiv 1$), L being therefore the length of m.

For such a curve m, we let $\theta(s)$ be the angle between the tangent to m at the point $m(s)$ and the horizontal axis, so that $\dot{m}_s = (\cos \theta, \sin \theta)$;

$$\theta : [0, L] \to [0, 2\pi[$$

will be called the angle function associated to m. A translation of m has no effect on θ, and an homothetie with factor λ simply transforms the function (θ:[0,L] $\to$ [0,2π[) into the function (θ_λ:[0,λL] $\to$ [0,2π[) with $\theta_\lambda(s) = \theta(s/\lambda)$. Thus a curve modulo translation and homothetie may be represented by its normalized angle function $\theta_{1/L}$, which is defined on [0,1]. In this paper, we will restrict ourselves to the scale invariant case, so we assume (without loss of generality) that all considered curves have length 1. A rotation of angle c acting on m transforms the function $\theta(.)$ into the function $\theta(.) + c$ (modulo 2π), or equivalently the function $e^{i\theta(.)}$ into the function $e^{ic}e^{i\theta(.)}$.

Letting $\zeta(.) = e^{i\theta(.)}$, we may thus represent curves by functions

$$\zeta : [0,1] \to S_1 = \{z \in \mathbb{C}, |z| = 1\},$$

this representation being translation and scale invariant. To add rotation invariance, we identify two functions ζ and $\tilde{\zeta}$ as soon as $\tilde{\zeta}/\zeta$ is constant, and denote by $[\zeta]$ the corresponding equivalence class.

Our matching will be based on a peculiar definition of deformations. If ϕ is an increasing diffeomorphism of [0,1], and r a complex function defined on [0,1] with values in S_1, we define the action

$$(\phi, r)\bigstar\zeta = r.\zeta \circ \phi.$$

The diffeomorphism ϕ (or more precisely its differential $\dot{\phi}_s$) may be interpreted as a tangential stretching of the curve, and $r(s)$ as a torsion of the curve at the point of arc-length s. One may show that the action can be interpreted as a group action if one sets

$$(\phi_1, r_1)\bigstar(\phi_2, r_2) = (\phi_2 \circ \phi_1, r_1.r_2 \circ \phi_1). \tag{8}$$

To be more specific, we consider the group G composed with couples (ϕ,r), such that

- ϕ:[0,1] $\to$ [0,1], $\phi(0) = 0$, $\phi(1) = 1$ and $\dot{\phi}_s > 0$ almost everywhere.
- r:[0,1] $\to \mathbb{C}$ measurable, and $|r(s)| = 1$ for all s. The identity in G is the pair $e = (\mathrm{id}, \mathbf{1})$ where id $(s) = s$ and $\mathbf{1}(s) = 1$ for all s. A small variation of e is of the kind $e + tX$ where X takes the form $X = (\xi,\rho)$ where ξ:[0,1] $\to \mathbb{R}$ and ρ:[0,1] $\to \mathbb{C}$ (and $\Re(\rho) = 0$). We define the cost of a deformation $m \mapsto (e + tX).m$ by $tN_e(X)$ where

$$N_e(\xi, \rho) = \sqrt{\int_0^1 (\dot{\xi}_s^2 + |\rho|^2) ds}.$$

This norm naturally arises from elementary considerations on small deformations of curves. It essentially corresponds to considering that the cost of the transformation $m \to m + V$ where V is a vector field on m is given by

$$\sqrt{\int_0^1 |\dot{V}_s|^2}.$$

In this particular case, the norm N does not depend on the deformed curve m. However, if we had considered the scale dependent case, it would have been necessary to add a dependance on the length of the deformed curve (see Ref. [5] for details). The term $\dot{\xi}_s^2$ penalizes *stretching* along the curve, whereas the term $|\rho|^2$ penalizes *torsion*.

We can formally apply the construction presented in Section 2.4. Note that it is much easier, and requires less differentiability assumptions, to differentiate Eq. (8) with respect to (ϕ_2,r_2) than with (ϕ_1,r_1). Thus there is a big advantage in using left-translations in the computations. Using Eq. (7), we obtain, after some computation,

$$W(\mathbf{g}) = \int_0^1 \sqrt{\int_0^1 \left(\frac{\ddot{\phi}_{ts}^2}{\dot{\phi}_s} + |\dot{r}_t|^2 \dot{\phi}_s \right) ds}$$

with $\mathbf{g}(t) = (\phi(t,.),r(t,.)), t \in [0,1]$.

If $g = (\phi,r) \in G$, we define

$$\Gamma(g) = \inf\{W(\mathbf{g}), \mathbf{g}(0) = e, \mathbf{g}(1) = g\}.$$

The following result is obtained in Ref. [5]

Theorem 1. *Let $g^* = (\phi^*,r^*) \in G$. We have*

$$\Gamma^* = \Gamma(\phi^*, r^*) = \arccos \int_0^1 \sqrt{\dot{\phi}_s^*} \left| \Re\left(\sqrt{r^*(s)} \right) \right| ds.$$

where $\Re(z)$ is the real part of the complex number z. Moreover, Γ^* is the cost of the path $\mathbf{g}(t) = (\phi(t,.),r(t,.))$ with

$$\dot{\phi}_s(t,s).r(t,s) = \left[\cos(\Gamma^* t) + \frac{\sqrt{\dot{\phi}_s^*(s) r^*(s)} - \cos\Gamma^*}{\sin\Gamma^*} \sin(\Gamma^* t) \right]^2 \tag{9}$$

The distance between two curves ζ and $\tilde{\zeta}$ is then given by

$$d(\zeta, \tilde{\zeta}) = \inf_{\phi} \arccos \int_0^1 \sqrt{\dot{\phi}_s} \left| \Re\left(\sqrt{\tilde{\zeta} \circ \phi(s)/\zeta(s)} \right) \right| ds. \tag{10}$$

It is proved in Ref. [5] that d is a true distance, and in particular that

$$d(\zeta, \tilde{\zeta}) = 0 \Rightarrow \zeta = \tilde{\zeta}.$$

If $\zeta(.) = e^{i\theta(.)}$ and $\tilde{\zeta}(.) = e^{i\tilde{\theta}(.)}$, the formula above also writes

$$d(\zeta, \tilde{\zeta}) = \inf_{\phi} \arccos \int_0^1 \sqrt{\dot{\phi}_s} \left| \cos\left(\frac{\tilde{\theta} \circ \phi - \theta}{2} \right) \right| ds.$$

To add rotation invariance, that is to compare ‘‘shapes’’ $[\zeta]$ and $[\tilde{\zeta}]$, we simply set, c varying in $[0,2\pi]$,

$$d([\zeta], [\tilde{\zeta}]) = \inf_{\phi, c} \arccos \int_0^1 \sqrt{\dot{\phi}_s} \left| \cos\left(\frac{\tilde{\theta} \circ \phi - \theta - c}{2} \right) \right| ds.$$

When the infimum is attained for some ϕ^* (and c^* in the rotation invariant case), we shall say that ϕ^* is an optimal matching for the shapes $[\zeta]$ and $[\tilde{\zeta}]$. Sufficient conditions for existence will be provided in a forthcoming theoretical paper, Ref. [10].

Theorem 1 implies that, when ϕ^* is an optimal matching, the whole *deformation path* between the shapes can be directly computed, yielding a minimal cost interpolation between the curves, given by

$$Z(t,.) = \mathbf{g}(t) \bigstar \zeta \tag{11}$$

where $\mathbf{g}$ is given by Eq. (9).

4. The case of closed curves

In the discussion above, we have assumed that the curves are given with arc-length parametrization defined on [0,1]. This implicitly assumes that a starting point and an end point have been selected on each curve. In the case of open curves, these points must coincide with the extremities, and if the curves are oriented, the parametrizations are uniquely defined (if the curves are not oriented, there are two possible choices). In the case of (oriented) closed curves, the starting point and the end point coincide and may be any point on the curves. The arc-length parametrizations are thus defined modulo a translation of the starting point along the curves. This should be taken into account in the matching, in which the cost function should be minimized over all choices for the origins of the curve parametrizations.

Another problem arises with the curve interpolation. Assume that two closed curves are matched, so that correct starting points are selected for the parametrizations. Let ζ and $\tilde{\zeta}$ be the complex representations of directions of the tangents; the fact that a curve is closed is equivalent to the identity $\int_0^1 \zeta(s)ds = 0$. However, the optimal interpolation $Z(t,.)$ defined in Eq. (11), does not have, in general, the property that $\int_0^1 Z(t,s)ds = 0$ for $t \in]0,1[$ (this, of course, is true for $t = 0$ and $t = 1$). This means that the optimal deformation path generally breaks closed curves. This is not a problem as far as matching is concerned, but is surely an undesirable feature for interpolation, since one would expect an interpolation of closed curves to be closed.

For this reason, we build a suboptimal interpolation path in which all curves remain closed. We want, however, this deformation path to be similar to the optimal one $Z(t,.)$, so we look, for all t, to the function $\tilde{Z}(t,.)$ which is the closest to $Z(t,.)$ with the constraint that for all t

$$\int_0^1 \tilde{Z}(t,s)ds = 0. \tag{12}$$

A consistent way to decide on the similarity of the functions $Z(t,.)$ and $\tilde{Z}(t,.)$ would be to use the distance d of Eq. (10). In this case, one should compute, for all t, the maximum over all ϕ and all $\tilde{Z}(t,.)$ satisfying the constraint Eq. (12), of

$$\int_0^1 \sqrt{\dot{\phi}_s} \left| \Re\left(\sqrt{\tilde{Z}(t,s)/Z(t,\phi(s))} \right) \right| ds.$$

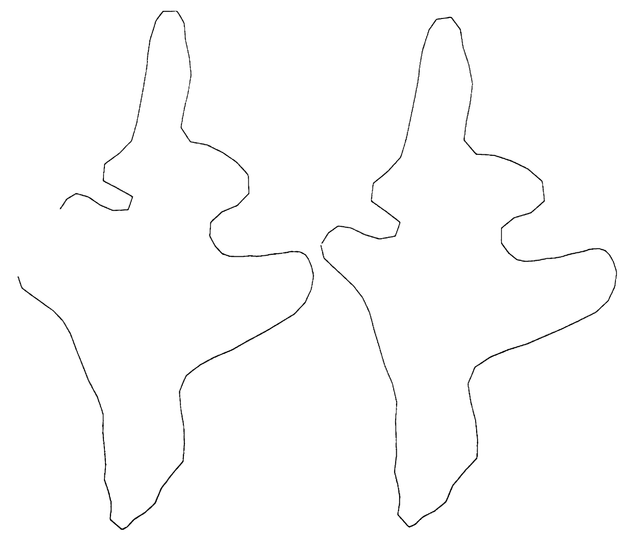

Fig. 1. An open curve (left) and the result of the closing operation (right).

This would, however, result in a very large computation cost, so it is preferable to make some approximations. We shall first impose $\phi = \mathrm{Id}$, so that, letting $Z(t,.) = \exp[i\alpha(t,.)]$ and $\tilde{Z}(t,.) = \exp[i\tilde{\alpha}(t,.)]$, the problem boils down to maximizing

$$\int_0^1 \left| \cos\left[\frac{\tilde{\alpha}(t,s) - \alpha(t,s)}{2} \right] \right| ds.$$

Using the fact that $1 - \cos(x/2) \simeq (1 - \cos x)/4$ and $\cos x > 0$ for small x, we get a further simplification if we maximize

$$\int_0^1 \cos[\tilde{\alpha}(t,s) - \alpha(t,s)]ds$$

with the constraints $\int_0^1 \cos \tilde{\alpha}(t,s)ds = \int_0^1 \sin \tilde{\alpha}(t,s)ds = 0.$

This problem becomes much more amenable to computation (recall that the unknown is $\tilde{\alpha}(t,.)$), and its solution provides an efficient and visually satisfying way for ''closing an open curve'' (cf. Fig. 1).

5. Numerical implementation

5.1. Case of polygonal curves

Given two functions θ and $\tilde{\theta}$, the core of the numerical problem is to compute

$$\sup_{\phi} \int_0^1 \sqrt{\dot{\phi}_s(s)} \left| \cos \frac{\tilde{\theta} \circ \phi(s) - \theta(s)}{2} \right| ds.$$

This is not trivial, since the functional to maximize is not concave, and not even differentiable because of the absolute value. The approach we use is to approximate the curves by polygons, for which some explicit computation may be carried out as we now demonstrate. So, assume that both

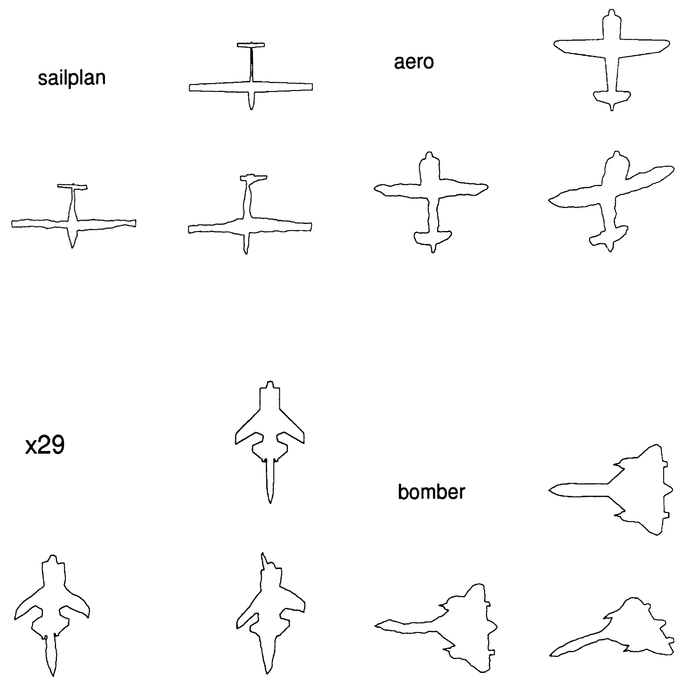

Fig. 2. Outlines of planes from 4 classes. For each type of plane: upper-right original view (from above); lower-left: view from above degraded by smooth noise; lower-right: slight variation of the angle of view and noise. The compared outlines are lower-left and lower-right.

curves are piecewise linear, i.e. that θ and $\tilde{\theta}$ are piecewise constant. We summarize here a computation which is detailed in Ref. [5]. We let

$$Q(\phi) = \int_0^1 \sqrt{\dot{\phi}_s(s)} \left| \cos \frac{\tilde{\theta} \circ \phi(s) - \theta(s)}{2} \right| ds.$$

There exists $0 = s_0 < s_1 < \cdots < s_m = 1$ (resp. $0 = \tilde{s}_0 < \tilde{s}_1 < \cdots < \tilde{s}_n = 1$) and constants $\theta_1,\ldots,\theta_m$ (resp. $\tilde{\theta}_1, \ldots, \tilde{\theta}_n$) such that $\theta(s) \equiv \theta_i$ on $[s_{i-1}, s_i[$(resp. $\tilde{\theta}(s) \equiv \tilde{\theta}_j$ on $[\tilde{s}_{j-1}, \tilde{s}_j[$).

Consider the points which match the s_i, that is $\tau_i = \phi^{-1}(s_i)$, $i = 1,\ldots,m$. One can completely characterize the best function ϕ when the τ_i are fixed (denote this function by ϕ_τ): for $i = 0,\ldots,m$, let $j(i)$ be the index j for which $\tau_i \in [\tilde{s}_j, \tilde{s}_{j+1}[$; one can show that

$$Q(\phi_\tau) = \sum_{i=1}^{m} \sqrt{(s_{i+1} - s_i) Q_i} \tag{13}$$

with

$$\begin{aligned} Q_i = {} & \cos^2 \frac{\tilde{\theta}_{j(i)+1} - \theta_{i+1}}{2} (\tilde{s}_{j(i)+1} - \tau_i) \\ & + \sum_{j=j(i)+1}^{j(i+1)-1} \cos^2 \frac{\tilde{\theta}_{j+1} - \theta_{i+1}}{2} (\tilde{s}_{j+1} - \tilde{s}_j) \\ & + \cos^2 \frac{\tilde{\theta}_{j(i+1)} - \theta_{i+1}}{2} (\tau_{i+1} - \tilde{s}_{j(i+1)}) \end{aligned}$$

We see that there exists a combinatorial part in the maximization of Q, which is due to the $j(i)$, $i = 1,\ldots, m$. Each $j(i)$ may take any value between 1 and n, with the constraint that $j(1) \le j(2) \le \cdots \le j(m)$. If the $j(i)$ are fixed, the τ_i may be obtained by the maximization of a smooth function, with the constraint that, for all i

$$\max(\tilde{s}_{j(i)}, \tau_{i-1}) \le \tau_i < \min(\tilde{s}_{j(i)+1}, \tau_{i+1}). \tag{14}$$

When the number of edges in the polygonal curve is not too

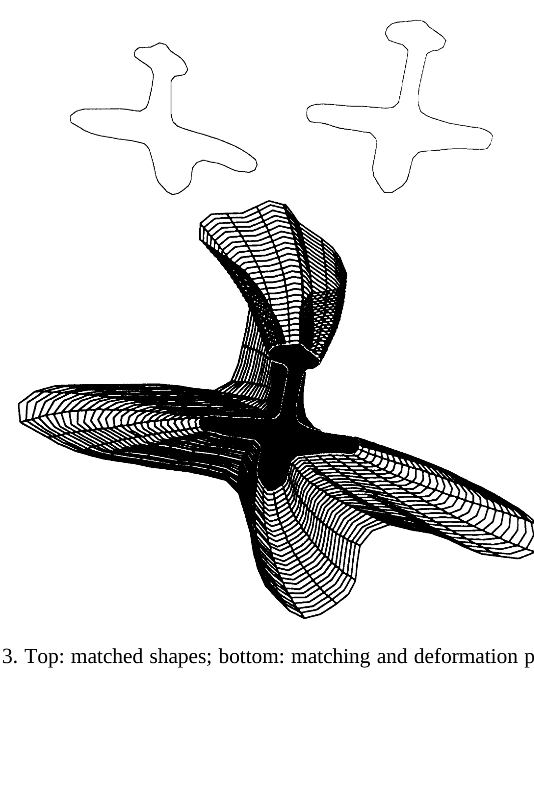

Fig. 3. Top: matched shapes; bottom: matching and deformation process.

Table 1
Matrix of distances (in radian) within the plane database (taken from Ref. [5])

	Sailplan-1	Sailplan-2	Aero-1	Aero-2	X29-1	X29-2	Bomber-1	Bomber-2
Sailplan-1	0.00	0.25	0.43	0.46	0.79	0.73	0.90	0.81
Sailplan-2	0.25	0.00	0.47	0.48	0.71	0.69	0.77	0.82
Aero-1	0.43	0.47	0.00	0.28	0.76	0.80	0.77	0.81
Aero-2	0.46	0.48	0.28	0.00	0.79	0.77	0.78	0.76
X29-1	0.79	0.71	0.76	0.79	0.00	0.38	0.84	0.81
X29-2	0.73	0.69	0.80	0.77	0.38	0.00	0.82	0.80
Bomber-1	0.90	0.77	0.77	0.78	0.84	0.82	0.00	0.29
Bomber-2	0.81	0.82	0.81	0.76	0.81	0.80	0.29	0.00

large, $Q(\phi_\tau)$, as given in Eq. (13), can be very quickly maximized by linear programming. When the number of edges is large, a suboptimal steepest-descent procedure may be used.

5.2. General curves

When dealing with general differentiable curves, each of them is replaced by a polygonal approximation. We generally use a multi-scale approach, starting with a rough polygonal approximation for which dynamic programming can be used, and then refine the result for enhanced approximations by steepest-descent.

To estimate the rotation parameter c in the expression of $d^{(0)}$, we start with an initial value c_0 and find the optimal g with this fixed c_0, and then compute the best c given g, iterating the procedure a few times.

6. Experimental results

6.1. Comparison

We present examples from a small database composed of 8 outlines of planes, for 4 types of planes. The shapes have been extracted from 3D synthetic images, under two slightly different view angles for each plane. We have applied some smooth stochastic noise to the outlines in order to obtain variants of the same shape which look more realistic. The outlines are presented in Fig. 2. The lengths of the curves have been computed after smoothing (using a cubic-spline representation). The complete matrix of distances has been computed from this database, and is given in Table 1. We see that the

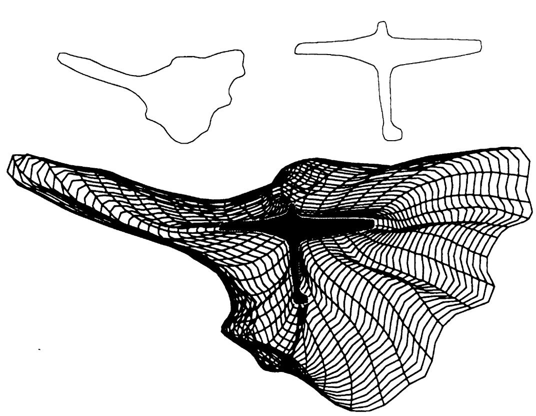

Fig. 4. Top: matched shapes; bottom: matching and deformation process.

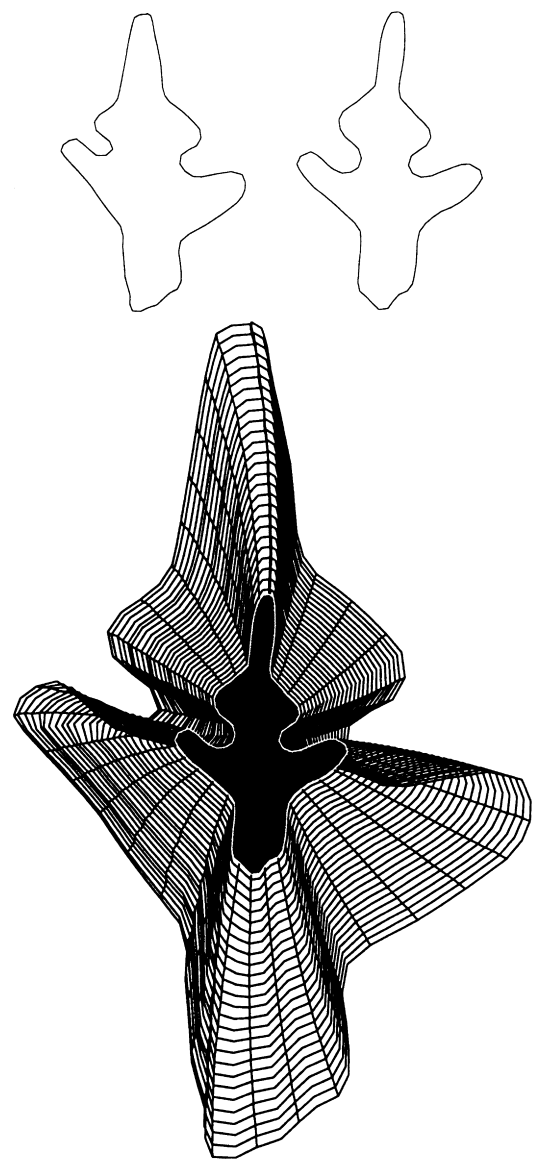

Fig. 5. Top: matched shapes; bottom: matching and deformation process. Deformation cost: 0.29.

distance between a plane and another one from the same class is always smaller than between any plane in a different class.

6.2. Matching

We show some computed matchings of silhouettes of planes, together with the associated interpolation paths (since these curves are closed, we have applied the procedure of Section 4 to perform the interpolation) Figs. 3–5. We draw in a single figure the whole deformation process, and, in order to make the visualisation possible, we continuously scale the deformed curves (note that we work on shapes so that we see curves independently of scales). The curves are also stacked one above the other in 3D-space, so that the output appears as a 3D surface seen from above. Transversal lines on this surface show the trajectory of each point during the process, ensuring that their endpoints provide optimal matching between the original shapes.

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