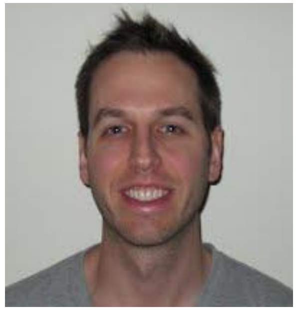
Is an Affine Constraint Needed for Affine Subspace Clustering?

Chong You (由翀) [1], Chun-Guang Li (李春光) [2], Daniel P. Robinson [3], Rene Vidal [3]

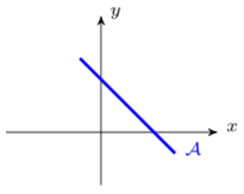




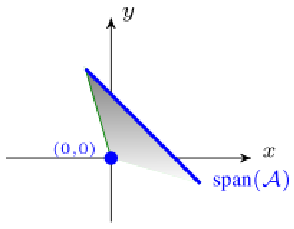

[1] University of California, Berkeley [2] 北京邮电大学
[3] Johns Hopkins University

Outline

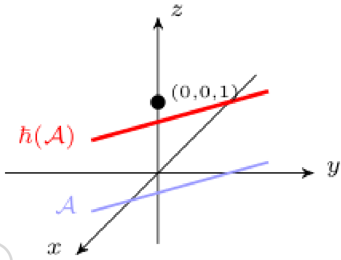
- Introduction
  - Subspace Clustering: what, why and how
- Background
  - 1. Self-Expression Model
  - 2. Self-Expression Model + Affine Constraint
- Our Contributions
  - Affine Subspace Clustering w/o Affine Constraint
- Empirical Results
  - Comparison on ASC with/without Affine Constraint
- Conclusions
  - When Is an Affine Constraint Needed?

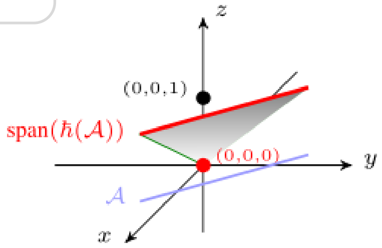


(a)



(b)



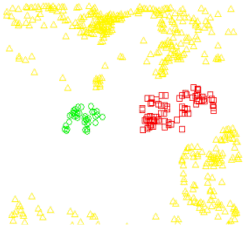


Structures in High-Dimensional Data

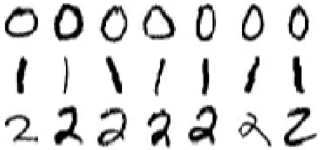
• High dimensional data in computer vision, pattern recognition, and bioinformatics are usually controlled by *a few hidden factors*

- Frontal face images of varying illumination condition

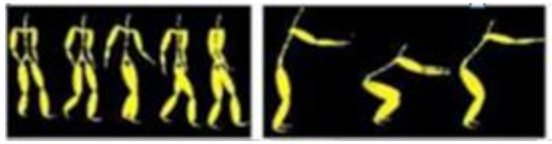
- Feature points trajectory on moving rigid object in video

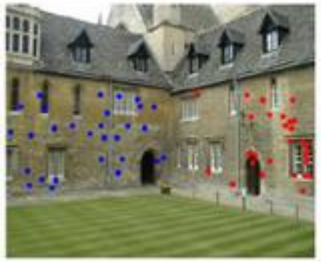


- Handwriting digit images

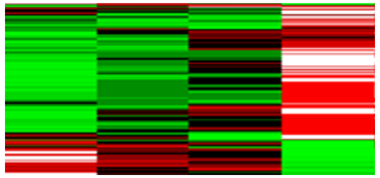


- Action sequence in a video





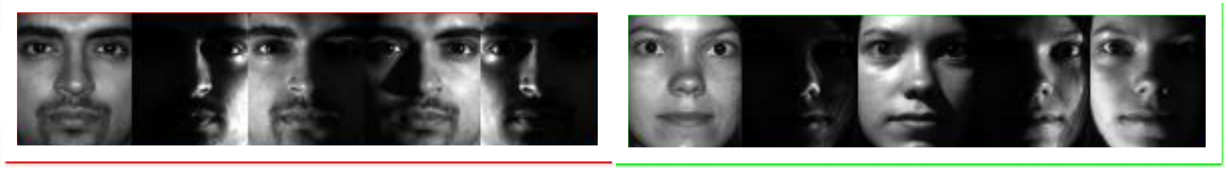
- Planar area in 3D vision



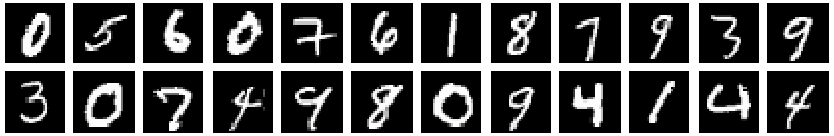
- Gene expression data of cancer subtype

Examples in Application

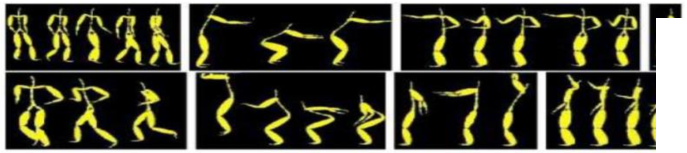
- Rigid Object Segmentation in Video



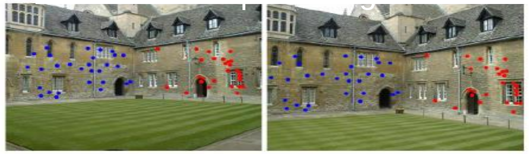
- Face Image Clustering



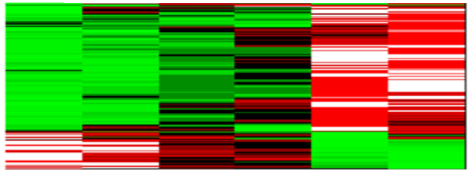
- Handwriting Digit Image Clustering



- Action Segmentation



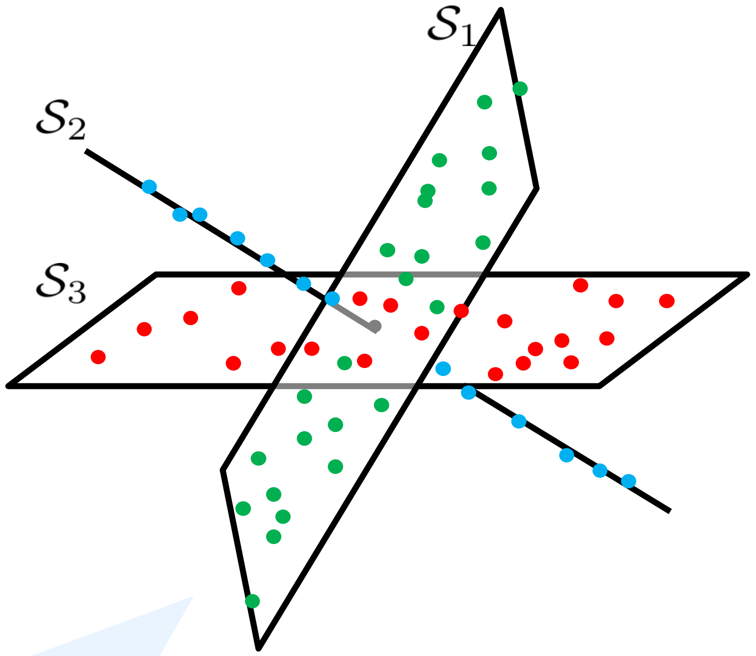
- Planar Area Detection in 3D



- Cancer Subtype Discovery

Examples in Application → Union of Subspaces

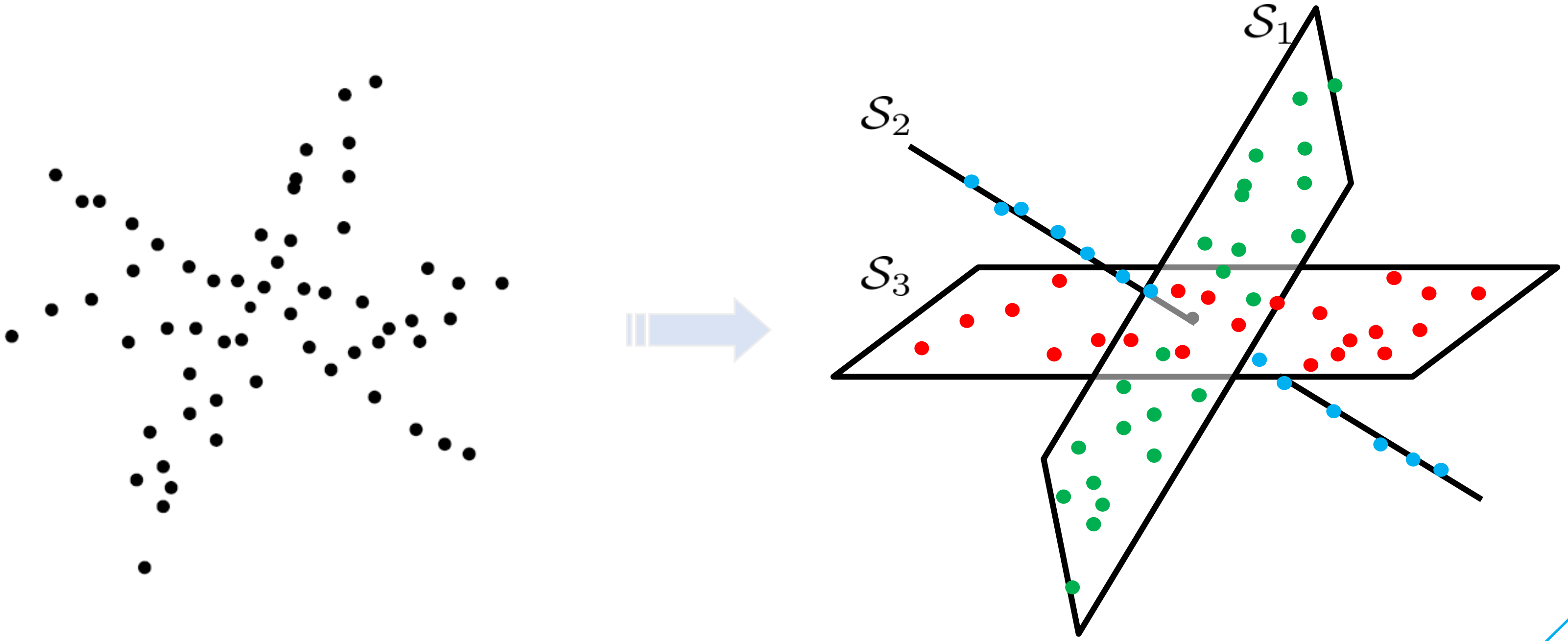
- Rigid Object Segmentation in Video
- Face Image Clustering
- Handwriting Digit Image Clustering
- Action Segmentation
- Planar Area Detection in 3D
- Cancer Subtype Discovery



Union of Subspaces

Subspace Clustering

- Given a set of data points lying (approximately) in a **union of subspaces**, to segment the data points into each subspace

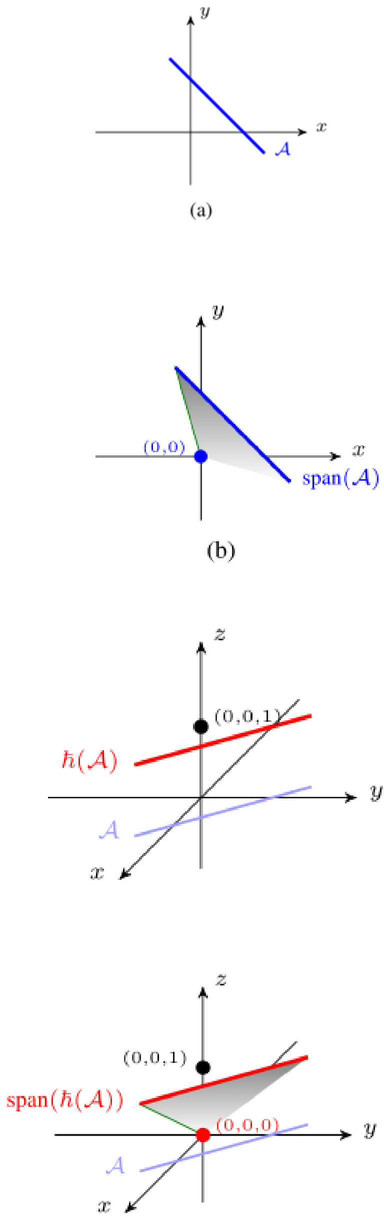


- ➢ Each subspace corresponds to a pattern (or cluster) in data
  - ✓ e.g. a moving object, person, digit, action, plan area, cancer subtype

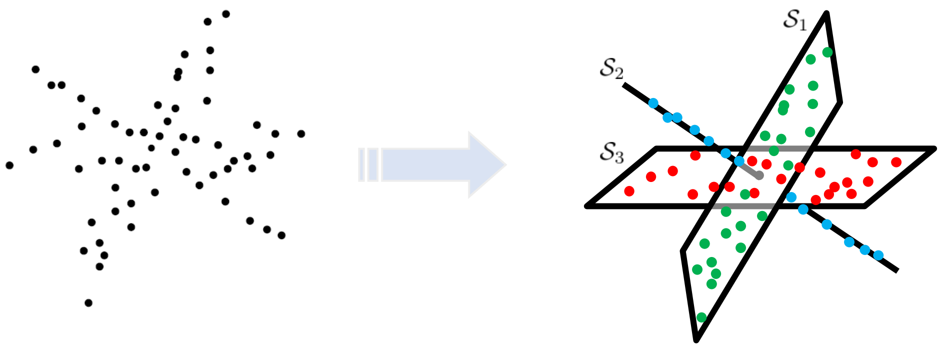
Outline

- Introduction
- Background
- Our Contributions
- Empirical Results
- Conclusions

1. Self-Expression Model
2. Self-Expression Model + Affine Constraint



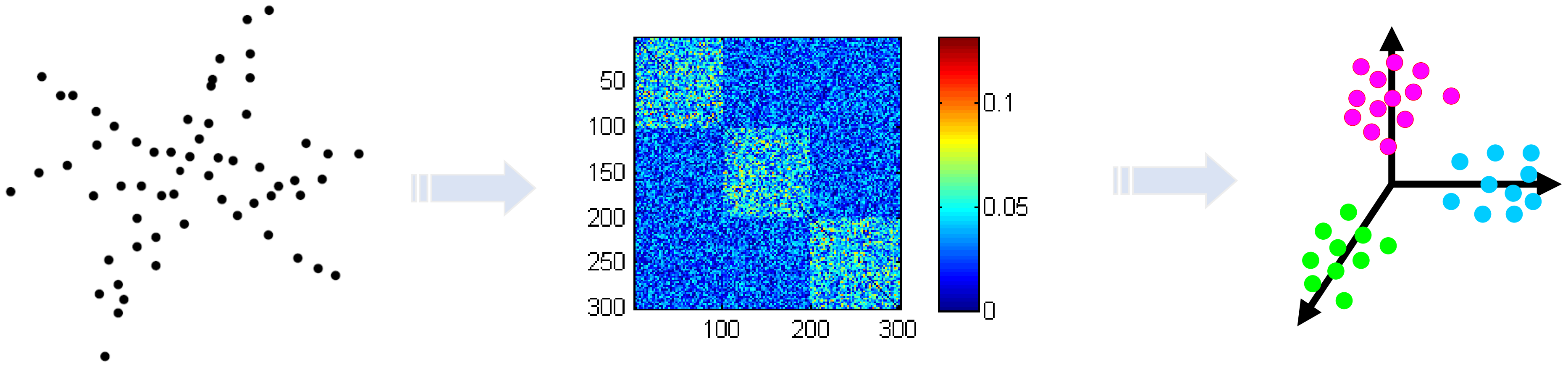
Subspace Clustering: Existing Methods



- Iterative methods
  - k-plane, q-flats, ⋯
- Statistical methods
  - Matrix Factorization, MPPCA, ⋯
- Algebraic methods
  - Generalized PCA (GPCA), ⋯
- Spectral Subspace Clustering
  - SSC (Elhamifar & Vidal: CVPR09), LRR (Liu et al. ICML10), LSR (Lu et al. ECCV12), SSSC(Peng et al. CVPR13), S^3C (Li & Vidal: CVPR15/ TIP17), EnSC (You et al. CVPR16), ESC(You et al. ECCV18)
  - Kernel SSC/Latent SSC(Vishal et al. ICCV13), Latent LRR(Liu et al. ICCV11), VR3(Peng et al. CVPR17), DSCNet (Ji et al.:NIPS17), DASC(Zhou et al.: CVPR18), S^2ConvSCN(Zhang et al. CVPR19)
  - Correntropy (He et al. TNNLS15), GMM / GP (Li et al.:CVPR15) / IPID(Li et al.: CVPR19)

Spectral Clustering based Methods

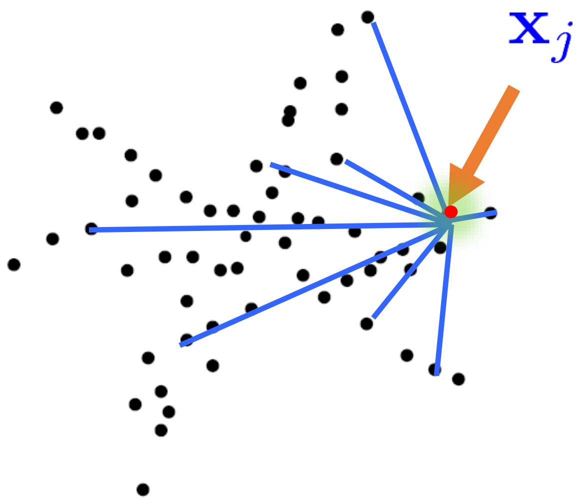
- Step 1: Build a Data Affinity Matrix
- Step 2: Apply Spectral Clustering



➢ Why spectral clustering? Spectral Graph Theory

Detection of Subspace Structures

- Step 1: Build a Data Affinity Matrix
- Step 2: Apply Spectral Clustering



- Self-Expression Model
  - Express a data point as a linear combination of other data points

$$\mathbf{x}_j = \sum_{i \neq j}^{N} c_i \mathbf{x}_i = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + c_{j-1} \mathbf{x}_{j-1} + 0 \cdot \mathbf{x}_j + c_{j+1} \mathbf{x}_{j+1} + \ldots + c_N \mathbf{x}_N$$

[1] E. Elhamifar & R. Vidal: "Sparse subspace clustering", CVPR 2009.
[2] E. Elhamifar & R. Vidal: "Sparse subspace clustering: Algorithm, theory, and applications", IEEE TPAMI 2013.

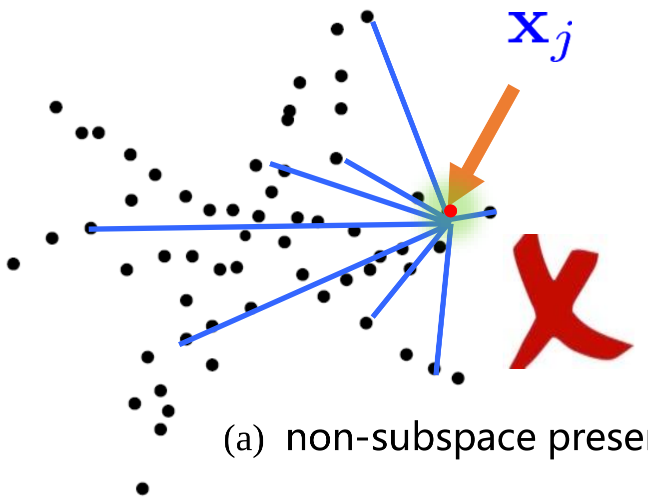
Subspace Preserving Property

- Self-Expression Model
  - For data points in a linear subspace, we have:

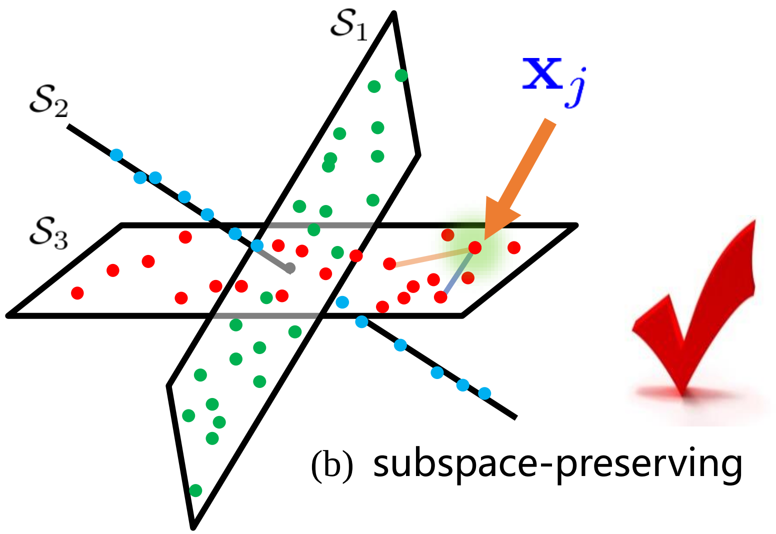
a linear combination

$$\mathbf{x}_j = \sum_{i \neq j}^{N} c_i \mathbf{x}_i = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + c_{j-1} \mathbf{x}_{j-1} + 0 \cdot \mathbf{x}_j + c_{j+1} \mathbf{x}_{j+1} + \ldots + c_N \mathbf{x}_N$$

- **Subspace-Preserving Property**: nonzero coefficients ONLY correspond to data points in the same subspace as x_j



(a) non-subspace preserving



(b) subspace-preserving

Subspace Preserving Property

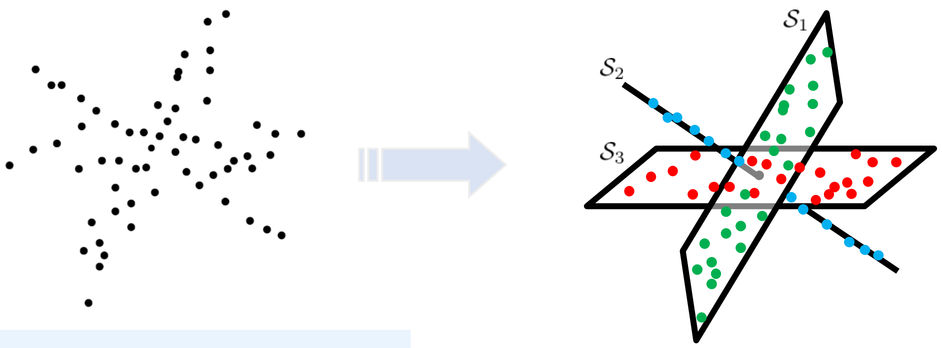
- Self-Expression Model
  - For data points in a linear subspace, we have:

a linear combination

$$\mathbf{x}_j = \sum_{i \neq j}^{N} c_i \mathbf{x}_i = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + c_{j-1} \mathbf{x}_{j-1} + 0 \cdot \mathbf{x}_j + c_{j+1} \mathbf{x}_{j+1} + \ldots + c_N \mathbf{x}_N$$

  - To find the coefficients with **subspace-preserving property**, a proper regularization is adopted, e.g.
    - ✓ $\|\mathbf{c}\|_1$ → Sparse Subspace Clustering (SSC)
    - ✓ $\|\mathbf{c}\|_2^2$ → Least Square Regression (LSR)
    - ✓ $\|C\|_*$ → Low-Rank Representation (LRR/LRSC)

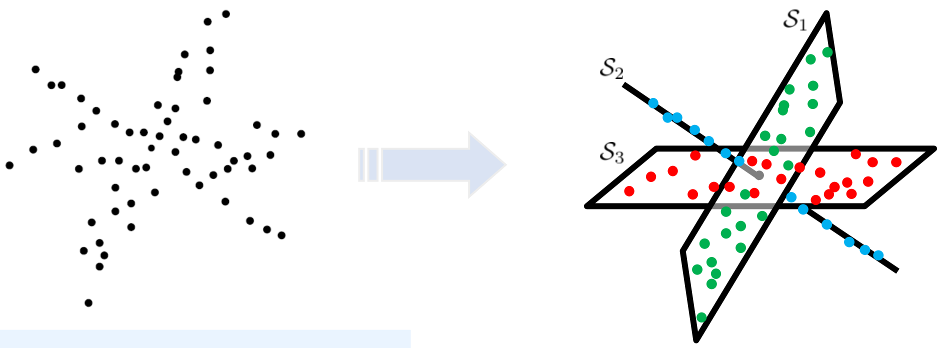
Subspace Clustering: Existing Methods



$$\min_{C,E} \|C\|_{\kappa} + \lambda \|E\|_{\ell} \quad \text{s.t. } X = XC + E,\ \operatorname{diag}(C) = 0$$

- Spectral Subspace Clustering
  - SSC (Elhamifar & Vidal: CVPR09), LRR (Liu et al. ICML10), LSR (Lu et al. ECCV12), SSSC(Peng et al. CVPR13), S^3C (Li & Vidal: CVPR15/ TIP17), EnSC (You et al. CVPR16), ESC(You et al. ECCV18)
  - Kernel SSC/Latent SSC(Vishal et al. ICCV13), Latent LRR(Liu et al. ICCV11), VR3(Peng et al. CVPR17), DSCNet (Ji et al.:NIPS17), DASC(Zhou et al.: CVPR18), S^2ConvSCN(Zhang et al. CVPR19)
  - Correntropy (He et al. TNNLS15), GMM / GP (Li et al.:CVPR15) / IPID(Li et al.: CVPR19)

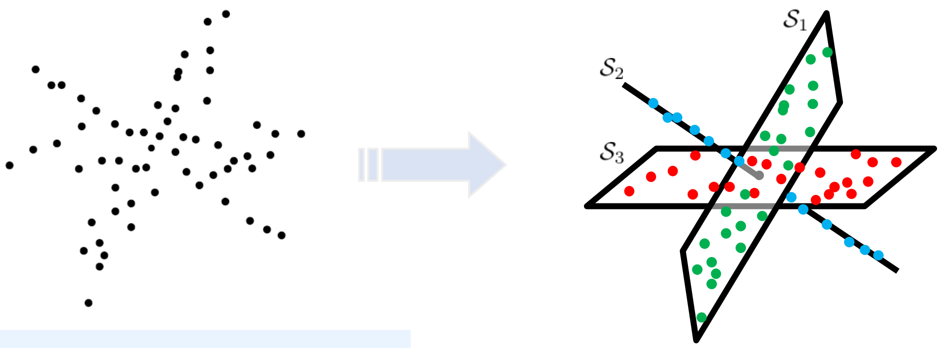
Subspace Clustering: Existing Methods



$$\min_{C,E} \|C\|_\kappa + \lambda \|E\|_\ell \quad \text{s.t.} \quad \varphi(X) = \varphi(X) C + E, \ \operatorname{diag}(C) = 0$$

- Spectral Subspace Clustering
  - SSC (Elhamifar & Vidal: CVPR09), LRR (Liu et al. ICML10), LSR (Lu et al. ECCV12), SSSC(Peng et al. CVPR13), S^3C (Li & Vidal: CVPR15/ TIP17), EnSC (You et al. CVPR16), ESC(You et al. ECCV18)
  - Kernel SSC/Latent SSC(Vishal et al. ICCV13), Latent LRR(Liu et al. ICCV11), VR3(Peng et al. CVPR17), DSCNet (Ji et al.:NIPS17), DASC(Zhou et al.: CVPR18), S^2ConvSCN(Zhang et al. CVPR19)
  - Correntropy (He et al. TNNLS15), GMM / GP (Li et al.:CVPR15) / IPID(Li et al.: CVPR19)

Subspace Clustering: Existing Methods



$$\min_{C,E} \|C\|_{\kappa} + \lambda \|E\|_{\ell} \quad \text{s.t.} \;\; X = XC + E, \; \text{diag}(C) = 0$$

- Spectral Subspace Clustering
  - SSC (Elhamifar & Vidal: CVPR09), LRR (Liu et al. ICML10), LSR (Lu et al. ECCV12), SSSC(Peng et al. CVPR13), S^{3}C (Li & Vidal: CVPR15/ TIP17), EnSC (You et al. CVPR16), ESC(You et al. ECCV18)
  - Kernel SSC/Latent SSC(Vishal et al. ICCV13), Latent LRR(Liu et al. ICCV11), VR3(Peng et al. CVPR17), DSCNet (Ji et al.:NIPS17), DASC(Zhou et al.: CVPR18), S^{2}ConvSCN(Zhang et al. CVPR19)
  - Correntropy (He et al. TNNLS15), GMM / GP (Li et al.:CVPR15) / IPID(Li et al.: CVPR19)

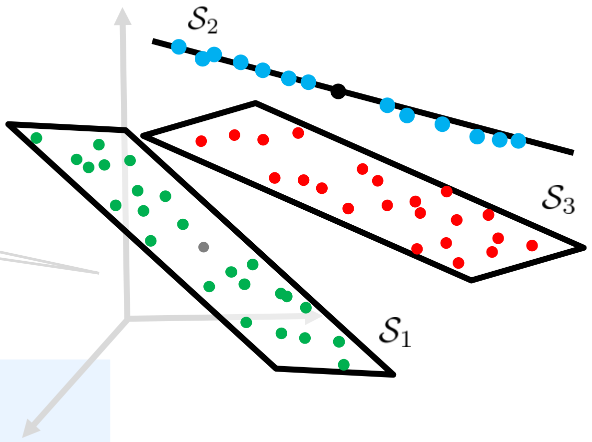
Subspace Clustering: Existing Methods

$$\min_{C,E} \|C\|_{\kappa} + \lambda \|E\|_{\ell} \quad \text{s.t.} \;\; X = XC + E, \; \mathrm{diag}(C) = 0$$
$$\mathbf{1}^T = \mathbf{1}^T C$$

- Spectral Subspace Clustering
  - SSC (Elhamifar & Vidal: CVPR09), LRR (Liu et al. ICML10), LSR (Lu et al. ECCV12), SSSC(Peng et al. CVPR13), S^3C (Li & Vidal: CVPR15/ TIP17), EnSC (You et al. CVPR16), ESC(You et al. ECCV18)
  - Kernel SSC/Latent SSC(Vishal et al. ICCV13), Latent LRR(Liu et al. ICCV11), VR3(Peng et al. CVPR17), DSCNet (Ji et al.:NIPS17), DASC(Zhou et al.: CVPR18), S^2ConvSCN(Zhang et al. CVPR19)
  - Correntropy (He et al. TNNLS15), GMM / GP (Li et al.:CVPR15) / IPID(Li et al.: CVPR19)

Subspace Clustering: Existing Methods

Union of Affine Subspaces



$$\min_{C,E} \|C\|_\kappa + \lambda \|E\|_\ell \quad \text{s.t.} \quad X = XC + E, \ \operatorname{diag}(C) = 0$$

$$\mathbf{1}^T = \mathbf{1}^T C$$

Affine Constraint → ?

- Spectral Subspace Clustering
  - SSC (Elhamifar & Vidal: CVPR09), LRR (Liu et al. ICML10), LSR (Lu et al. ECCV12), SSSC(Peng et al. CVPR13), S^3C (Li & Vidal: CVPR15/ TIP17), EnSC (You et al. CVPR16), ESC(You et al. ECCV18)
  - Kernel SSC/Latent SSC(Vishal et al. ICCV13), Latent LRR(Liu et al. ICCV11), VR3(Peng et al. CVPR17), DSCNet (Ji et al.:NIPS17), DASC(Zhou et al.: CVPR18), S^2ConvSCN(Zhang et al. CVPR19)
  - Correntropy (He et al. TNNLS15), GMM / GP (Li et al.:CVPR15) / IPID(Li et al.: CVPR19)

Affine Self-Expression Model

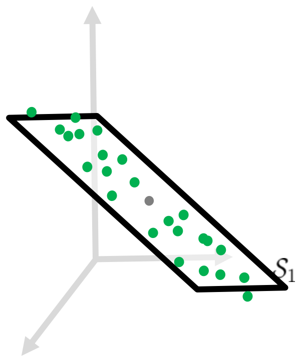
- Affine Self-Expression Model
  - For data points in an affine subspace, we have

an affine combination

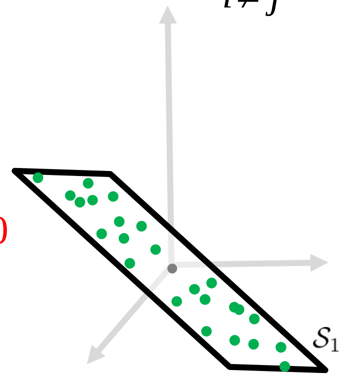
$$\mathbf{x}_j = \sum_{i \neq j}^{N} c_i \mathbf{x}_i = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + c_{j-1} \mathbf{x}_{j-1} + 0 \cdot \mathbf{x}_j + c_{j+1} \mathbf{x}_{j+1} + \ldots + c_N \mathbf{x}_N \quad \text{s.t.} \sum_{i=1}^{N} c_i = 1$$

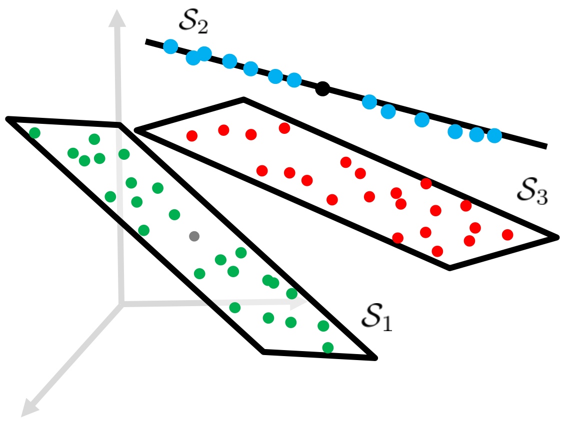
✓ Affine constraint $\sum_{i=1}^{N} c_i = 1 \rightarrow$ Coefficients are translation invariant.

$$\mathbf{x}_j - \mathbf{x}_0 = \sum_{i \neq j}^{N} c_i \left(\mathbf{x}_i - \mathbf{x}_0 \right) = \sum_{i \neq j}^{N} c_i \mathbf{x}_i - \mathbf{x}_0 \sum_{i \neq j}^{N} c_i = \sum_{i \neq j}^{N} c_i \mathbf{x}_i - \mathbf{x}_0$$



translate $\mathbf{x}_0$





[1] E. Elhamifar & R. Vidal: “Sparse subspace clustering”, CVPR 2009.

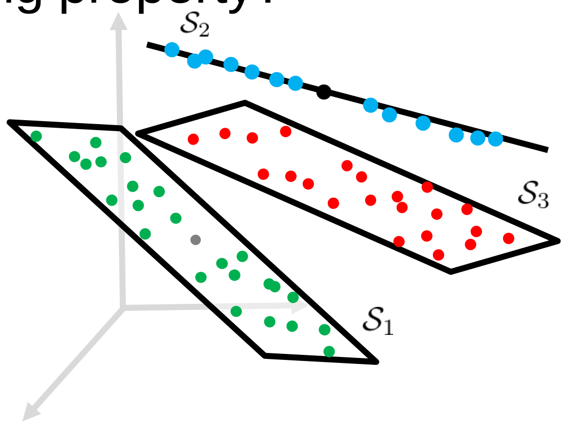
Affine Self-Expression Model

- Affine Self-Expression Model
  - For data points in an affine subspace, we have

$$\mathbf{x}_j = \sum_{i \neq j}^{N} c_i \mathbf{x}_i = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + c_{j-1} \mathbf{x}_{j-1} + 0 \cdot \mathbf{x}_j + c_{j+1} \mathbf{x}_{j+1} + \ldots + c_N \mathbf{x}_N \quad \text{s.t.} \sum_{i=1}^{N} c_i = 1$$

an affine combination

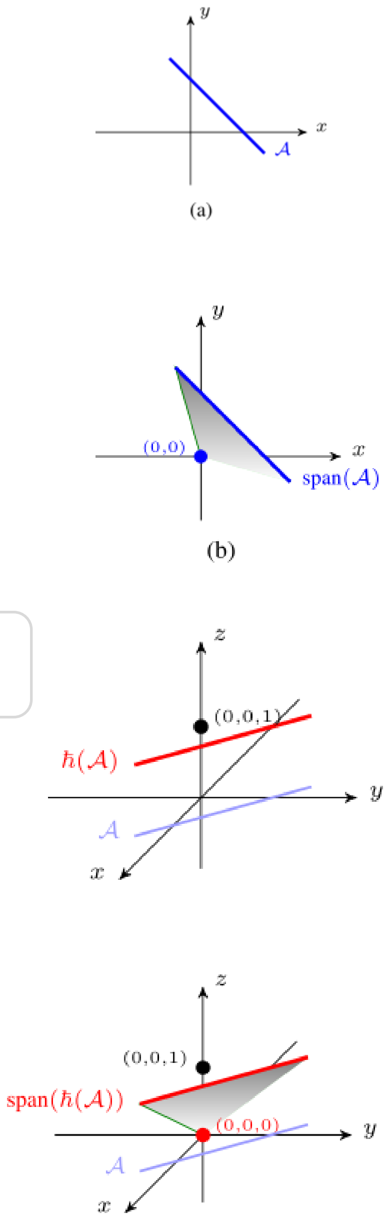
  - To find the coefficients with subspace-preserving property, a proper regularization is adopted, e.g. $\|\mathbf{c}\|_1$, $\|\mathbf{c}\|_2^2$, $\|C\|_*$.
    - Can it still find self-expressive coefficients with subspace-preserving property?
    - If YES, under what conditions?
    - Is an affine constraint necessary for affine subspace clustering?



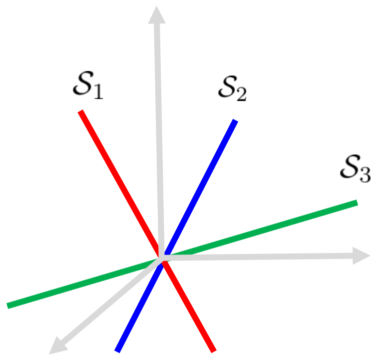
Outline

- Introduction
- Background
- Our Contributions
- Empirical Results
- Conclusions

Affine Subspace Clustering w/o Affine Constraint



Arrangement of Subspaces



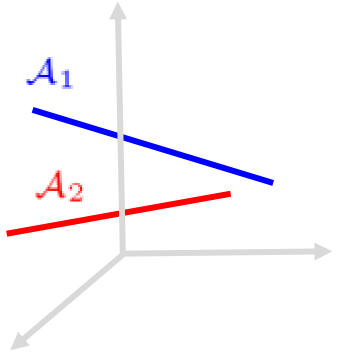
- Linearly independent vs. affinely independent

➤ Definition 1: The linear subspaces $\{\mathcal{S}_\ell\}_{\ell=1}^n$ (linearly) independent if

$$dim(span(\cup_{\ell=1}^n \mathcal{S}_\ell)) = \sum_{\ell=1}^n dim(\mathcal{S}_\ell)$$

➤ Definition 2: The affine subspaces $\{\mathcal{A}_\ell\}_{\ell=1}^n$ affinely independent if

$$\dim(\mathrm{aff}(\cup_{\ell=1}^n \mathcal{A}_\ell)) + 1 = \sum_{\ell=1}^n \dim(\mathcal{A}_\ell) + n$$



[1] C.-G. Li, C. You, and R. Vidal, "On Geometric Analysis of Affine Sparse Subspace Clustering", IEEE J. STSP, Vol.12, Issue 6, pp.1520-1533, Dec. 2018.

Affine Subspace Clustering with Affine Constraint

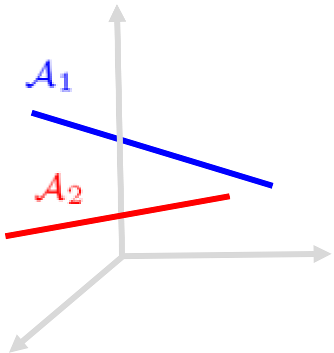
- Affine Subspace Clustering (ASC):

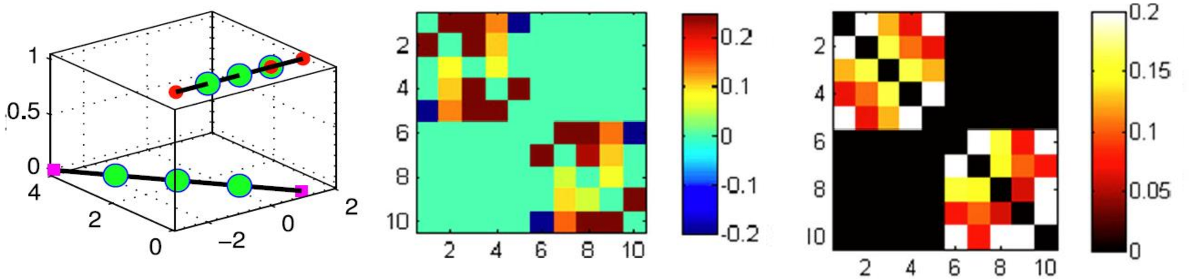
$$\min_{\mathbf{c}} r(\mathbf{c}) \ \text{ s.t. } \mathbf{x}_j = X\mathbf{c},\ c_j = 0,\ \mathbf{1}^T\mathbf{c} = 1 \quad (1)$$

where r(·) satisfies the Extended Block Diagonal condition (EBD)

function $f:\Omega \to \mathbb{R}$ *is said to satisfy the EBD conditions if*
- Ω *is closed under permutations and* f *is permutation invariant, i.e., for any* $\mathbf{C} \in \Omega$ *we have* $\mathbf{P}^\top\mathbf{C}\mathbf{P} \in \Omega$ *and* $f(\mathbf{C}) = f(\mathbf{P}^\top\mathbf{C}\mathbf{P})$ *for any permutation matrix* $\mathbf{P}$, *and*
- *for any partition* $\mathbf{C} = \begin{bmatrix}\mathbf{C}_1 & \mathbf{C}_3\\ \mathbf{C}_4 & \mathbf{C}_2\end{bmatrix}$ *of any matrix* $\mathbf{C} \in \Omega$ *such that* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *are square matrices we have* $\begin{bmatrix}\mathbf{C}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{C}_2\end{bmatrix} \in \Omega$ *and* $f\left(\begin{bmatrix}\mathbf{C}_1 & \mathbf{C}_3\\ \mathbf{C}_4 & \mathbf{C}_2\end{bmatrix}\right) \geq f\left(\begin{bmatrix}\mathbf{C}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{C}_2\end{bmatrix}\right)$ *with equality holding if and only if* $\mathbf{C}_3 = \mathbf{C}_4 = \mathbf{0}$.

➢ Theorem 1: If the affine subspaces $\{\mathcal{A}_\ell\}_{\ell=1}^n$ **affinely independent**, any optimal solution to (1) is subspace-preserving.





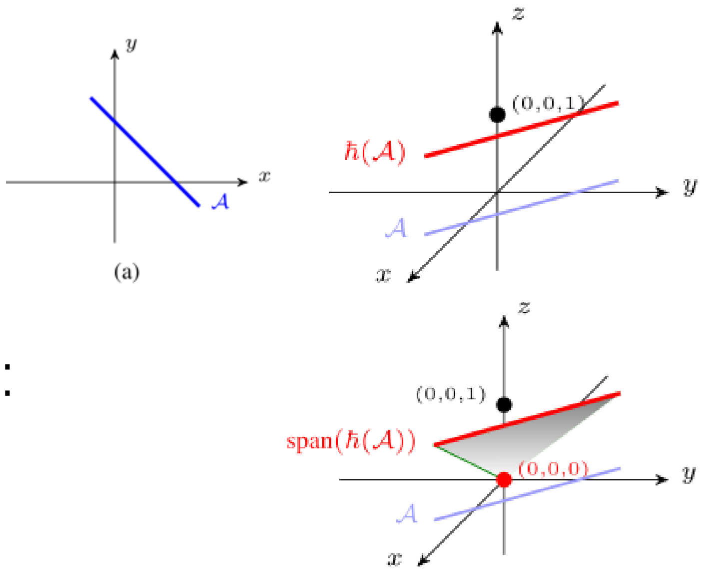
(a) data points (b) coefficients matrix C (c) affinity matrix A

[1] C. You, C.-G. Li, R. Daniel, R. Vidal, “Is an affine constraint needed for affine subspace clustering?”, accepted by ICCV2019.

Affine Subspace Clustering with Affine Constraint

- ASC with Affine Constraint:

$$\min_{\mathbf{c}} r(\mathbf{c}) \ \text{s.t.}\ \mathbf{x}_j = X\mathbf{c},\ c_j = 0,\ \mathbf{1}^T\mathbf{c} = 1 \quad (1)$$



➢ Reformulate (1) with homogenous embedding:

$$\min_{\mathbf{c}} r(\mathbf{c}) \ \text{s.t.} \begin{bmatrix} \mathbf{x}_j \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ \mathbf{1}^T \end{bmatrix} \mathbf{c}, \quad c_j = 0$$

✓ Theorem 2: If the linear subspaces $\{span(\hbar(\mathcal{A}_\ell))\}_{\ell=1}^n$ linearly) independent, if and only if affine subspaces are affinely $\{\mathcal{A}_\ell\}_{\ell=1}^n$ ndent

[1] C. You, C.-G. Li, R. Daniel, R. Vidal, "Is an affine constraint needed for affine subspace clustering?", accepted by ICCV2019.

Affine Subspace Clustering with Affine Constraint

- ASC with Affine Constraint:

$$\min_{\mathbf{c}} r(\mathbf{c}) \ \text{ s.t. } \mathbf{x}_j = X\mathbf{c},\ c_j = 0,\ \mathbf{1}^T\mathbf{c} = 1 \quad (1)$$

➢ Random Affine Subspace Model:

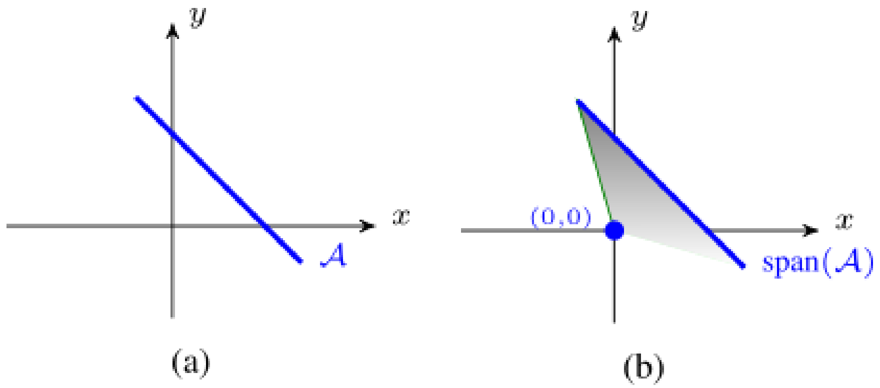
$$\mathcal{A}_\ell = \boldsymbol{w}_{0,\ell} + span\{\boldsymbol{w}_{1,\ell}, \cdots, \boldsymbol{w}_{d_\ell,\ell}\}$$

- where $\{\boldsymbol{w}_{0,\ell}, \boldsymbol{w}_{1,\ell}, \cdots, \boldsymbol{w}_{d_\ell,\ell}\}_{\ell=1}^n$ re independently and uniformly drawn from unit sphere of $\mathbb{R}^D$

✓ Theorem 3: If affine subspaces $\{\mathcal{A}_\ell\}_{\ell=1}^n$ generated with the random model and $D \geq \sum_{\ell=1}^n d_\ell + n - 1$ are $\{\mathcal{A}_\ell\}_{\ell=1}^n$ independent with probability 1 .

[1] C. You, C.-G. Li, R. Daniel, R. Vidal, "Is an affine constraint needed for affine subspace clustering?", accepted by ICCV2019.

Affine Subspace Clustering without Affine Constraint



- ASC without Affine Constraint:

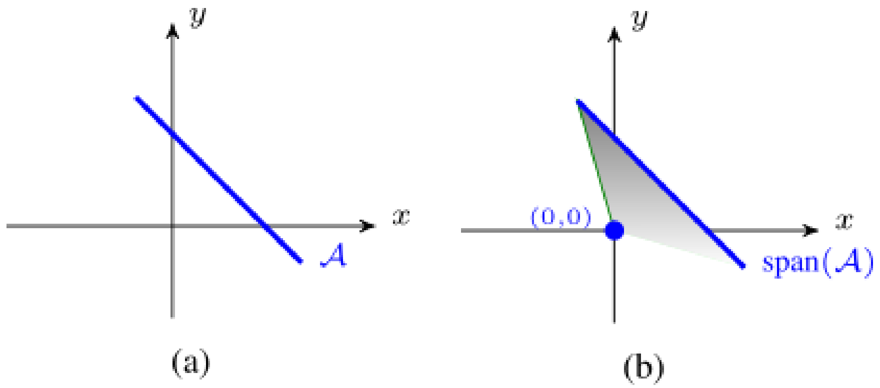
$$\min_{\mathbf{c}} r(\mathbf{c}) \;\; \text{s.t. } \mathbf{x}_j = X\mathbf{c},\; c_j = 0 \qquad (2)$$

where r(·) satisfies the Extended Block Diagonal condition (EBD)

➢ Theorem 4: If (linear) subspaces $\{span(\mathcal{A}_\ell)\}_{\ell=1}^{n}$ linearly independent, any optimal solution to (2) is subspace-preserving.

[1] C. You, C.-G. Li, R. Daniel, R. Vidal, "Is an affine constraint needed for affine subspace clustering?", accepted by ICCV2019.

Affine Subspace Clustering without Affine Constraint



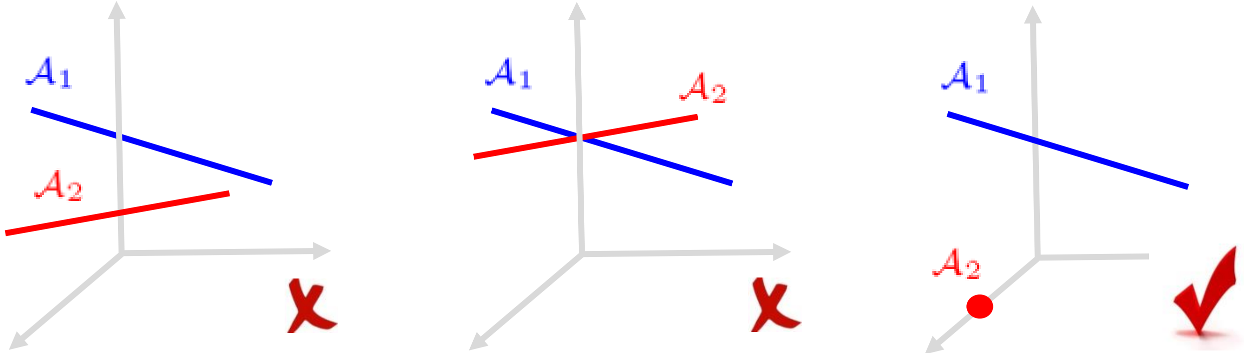
- ASC without Affine Constraint:

$$\min_{\mathbf{c}} r(\mathbf{c}) \text{ s.t. } \mathbf{x}_j = X\mathbf{c},\ c_j = 0 \quad (2)$$

where r(·) satisfies the Extended Block Diagonal condition (EBD)

➤ Theorem 4: If (linear) subspaces $\{span(\mathcal{A}_\ell)\}_{\ell=1}^n$ linearly independent, any optimal solution to (2) is subspace-preserving.

✓ Proposition 1: $\{span(\mathcal{A}_\ell)\}_{\ell=1}^n$ *e linearly independent* ***if and only if*** *:* $\mathbf{0} \notin aff(\cup_{\ell=1}^n \mathcal{A}_\ell)$ *d* $\{\mathcal{A}_\ell\}_{\ell=1}^n$ *are* ***affinely*** *independent.*

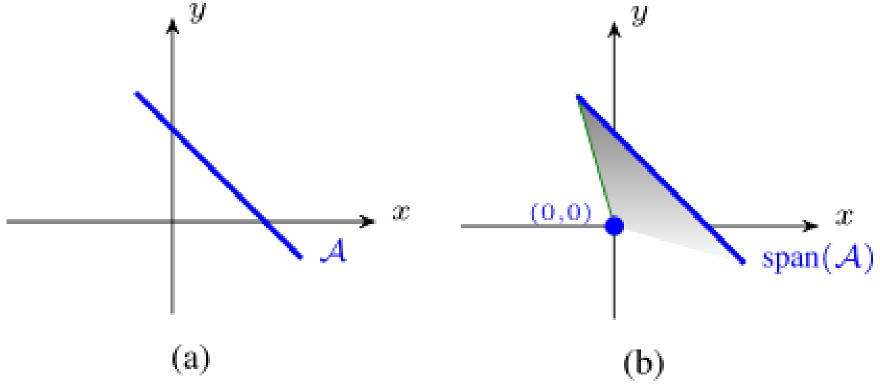


[1] C. You, C.-G. Li, R. Daniel, R. Vidal, "Is an affine constraint needed for affine subspace clustering?", accepted by ICCV2019.

Affine Subspace Clustering without Affine Constraint

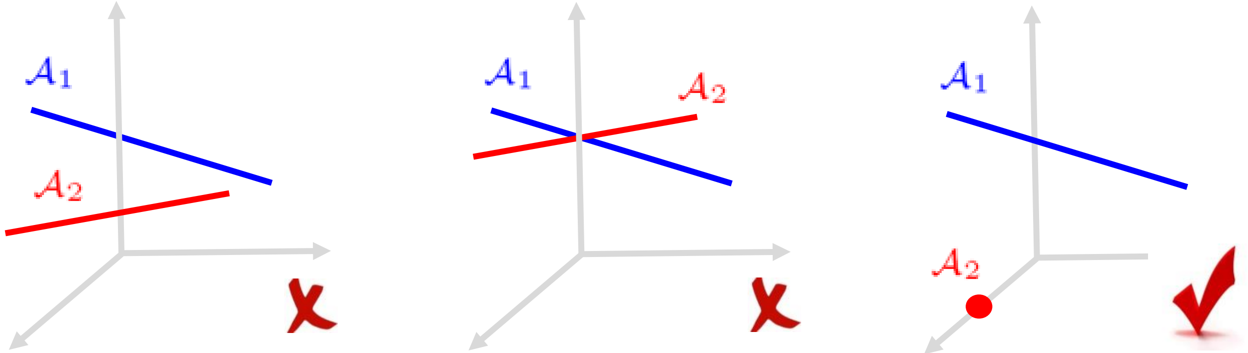
- ASC without Affine Constraint:

$$\min_{\mathbf{c}} r(\mathbf{c}) \quad \text{s.t. } \mathbf{x}_j = X\mathbf{c},\ c_j = 0 \qquad (2)$$



where r(·) satisfies the Extended Block Diagonal condition (EBD)

> Theorem 5: If affine subspaces $\{\mathcal{A}_\ell\}_{\ell=1}^n$ generated with the random model and $D \geq \sum_{\ell=1}^n d_\ell + n$ $\mathbf{0} \notin \mathit{aff}(\cup_{\ell=1}^n \mathcal{A}_\ell)$ are $\{\mathcal{A}_\ell\}_{\ell=1}^n$ independent with probability 1 .



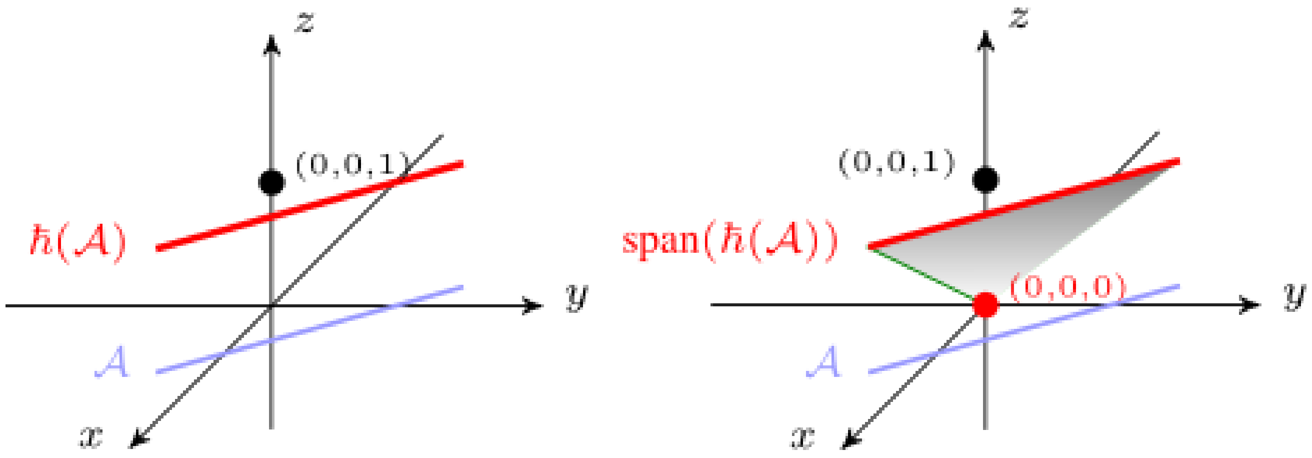
[1] C. You, C.-G. Li, R. Daniel, R. Vidal, “Is an affine constraint needed for affine subspace clustering?”, accepted by ICCV2019.

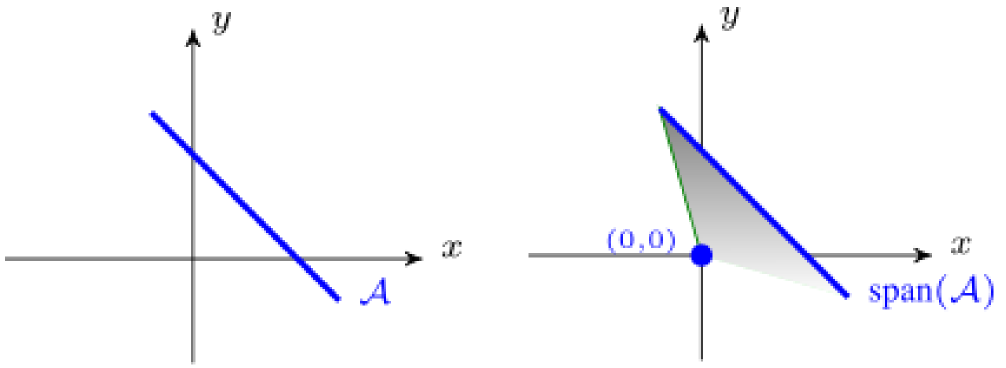
ASC *without or with* Affine Constraint

- $$\min_{\mathbf{c}} r(\mathbf{c}) \;\; \text{s.t. } \mathbf{x}_j = X\mathbf{c},\; c_j = 0,\; \mathbf{1}^T\mathbf{c} = 1 \qquad (1)$$
  - ➢ $\{\mathcal{A}_\ell\}_{\ell=1}^n$ are affinely independent
  - ➢ Under the random model, if
    $$D \geq \sum_{\ell=1}^n d_\ell + n - 1$$
    then every optimal solution to (1) is subspace preserving

$$\min_{\mathbf{c}} r(\mathbf{c}) \;\; \text{s.t. } \mathbf{x}_j = X\mathbf{c},\; c_j = 0 \qquad (2)$$

- ➢ $\{\mathcal{A}_\ell\}_{\ell=1}^n$ are affinely independent and $\mathbf{0} \notin aff(\cup_{\ell=1}^n \mathcal{A}_\ell)$
- ➢ Under the random model, if
  $$D \geq \sum_{\ell=1}^n d_\ell + n$$
  then every optimal solution to (2) is subspace preserving





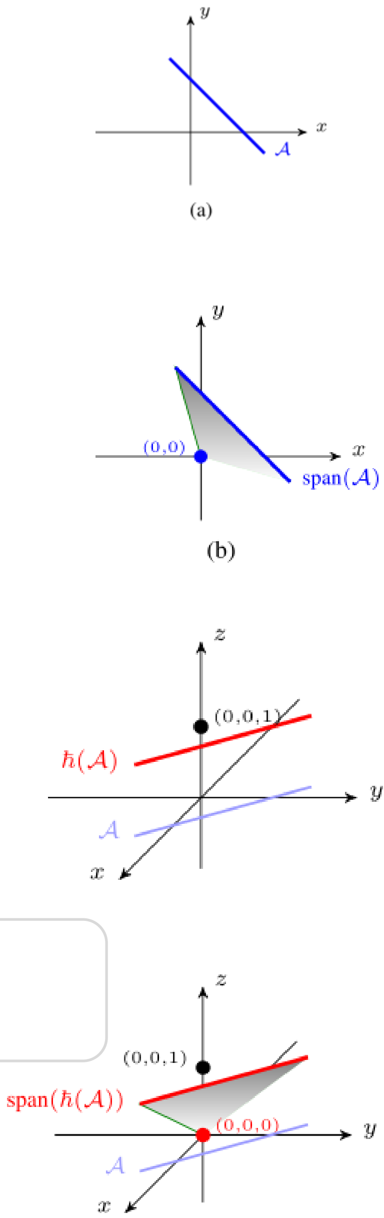
(a) (b)

Outline

- Introduction
- Background
- Our Contributions
- Empirical Results

Comparison on ASC with/without Affine Constraint

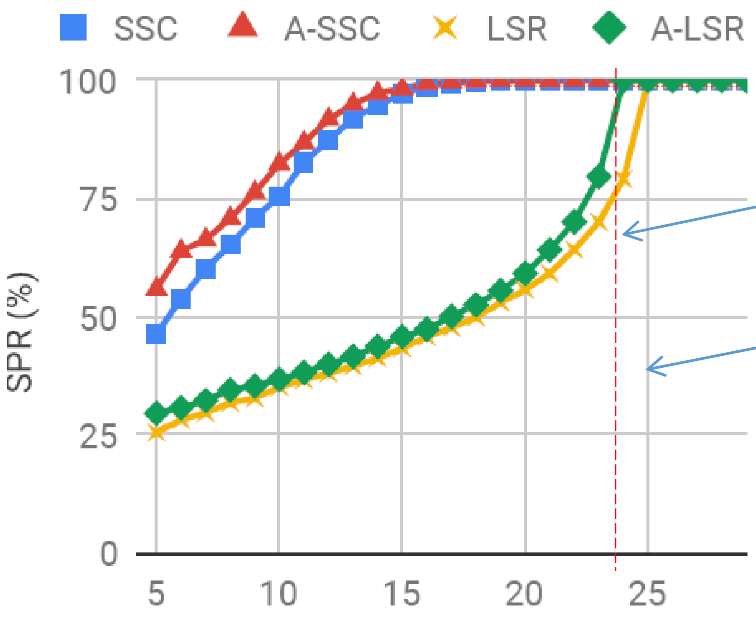
- Conclusions



ASC with or without Affine Constraint

- $$\min_{\mathbf{c}} r(\mathbf{c}) \ \text{ s.t. } \mathbf{x}_j = X\mathbf{c},\ c_j = 0,\ \mathbf{1}^T\mathbf{c} = 1 \quad (1) \qquad\qquad \min_{\mathbf{c}} r(\mathbf{c}) \ \text{ s.t. } \mathbf{x}_j = X\mathbf{c},\ c_j = 0, \quad (2)$$

➢ **Synthetic Data**: Generate n=5 affine subspaces under the random model, each affine subspace with dimension d=4.



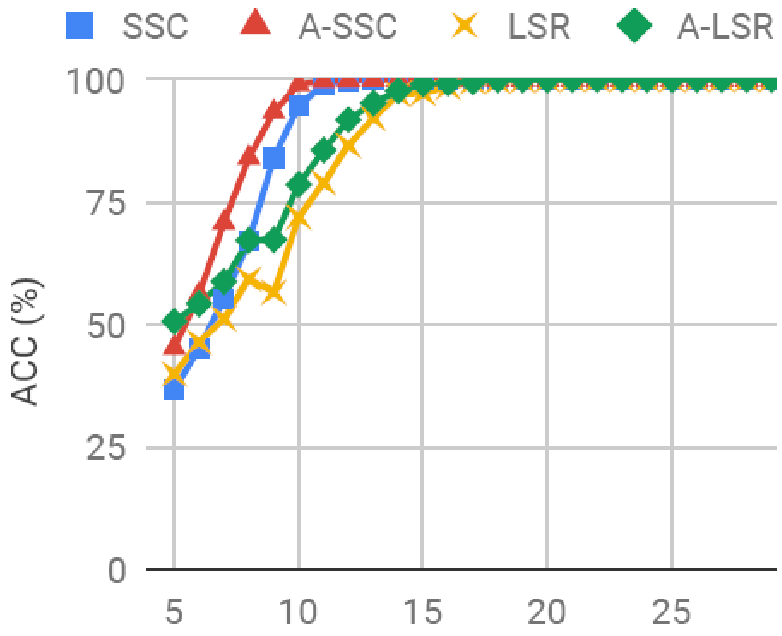
(a) SPR vs. D

$$D \geq \sum_{\ell=1}^{n} d_\ell + n - 1$$

$= 24$

$$D \geq \sum_{\ell=1}^{n} d_\ell + n$$

$= 25$



(b) ACC vs. D

ASC with or without Affine Constraint

- $$\min_{\mathbf{c}} r(\mathbf{c}) + \frac{\lambda}{2}\|\mathbf{x}_j - X\mathbf{c}\|_2^2 \quad \text{s.t. } c_j = 0,\ \mathbf{1}^T\mathbf{c} = 1 \quad (1) \qquad \min_{\mathbf{c}} r(\mathbf{c}) + \frac{\lambda}{2}\|\mathbf{x}_j - X\mathbf{c}\|_2^2 \quad \text{s.t. } c_j = 0 \quad (2)$$

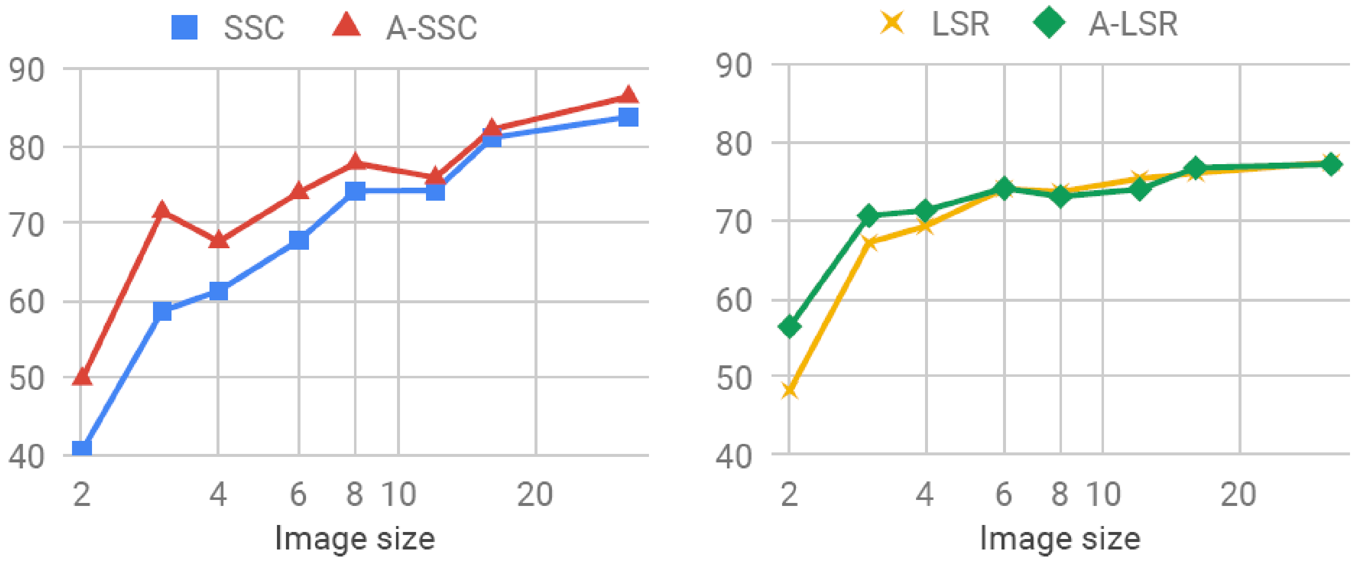


Figure 4. Performance on Coil-100 with various ambient dimension. Images in Coil-100 are downsampled to size p-by-p, with p varied on the x-axis, and clustering accuracy shown on the y-axis.

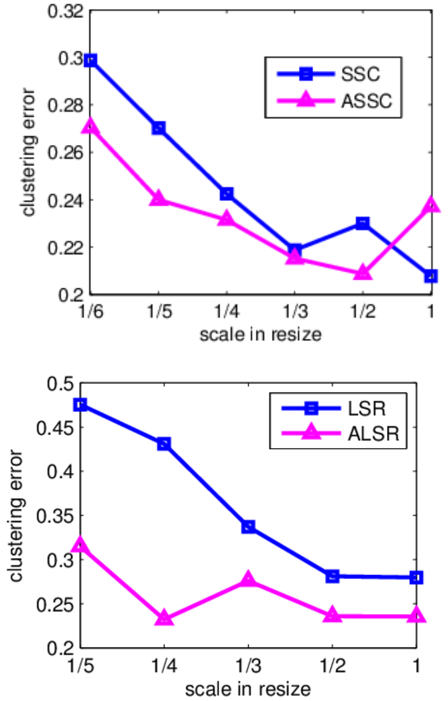
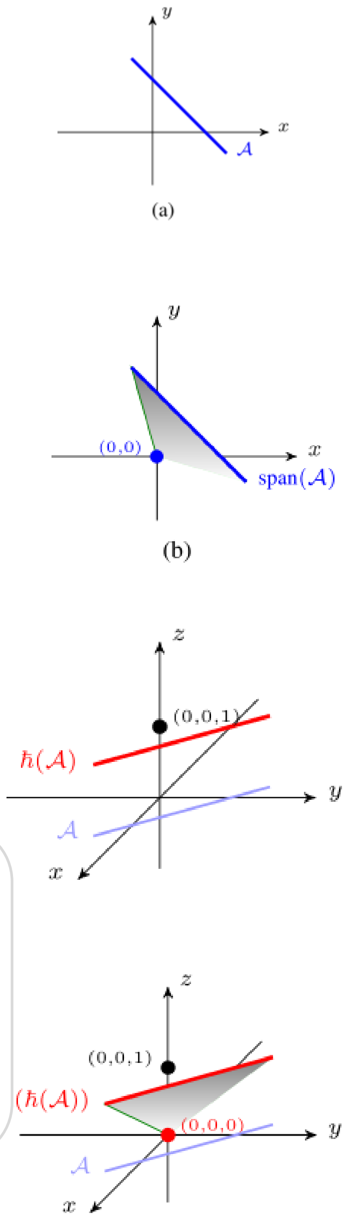


Figure 5. Performance on ORL400 with various ambient dimension

Outline

- Introduction
- Background
- Our Contributions
- Empirical Results
- Conclusions

When the dimension D of ambient space is lower, the affine constraint is needed.

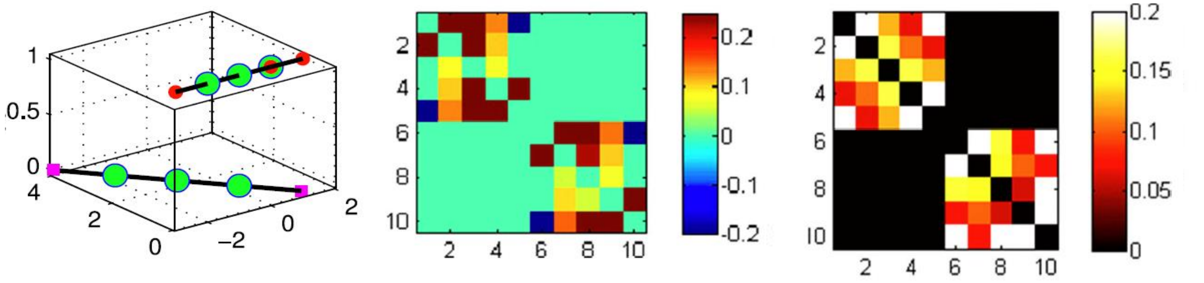


On Geometric Analysis of Affine SSC (1/3)

- Affine SSC (ASSC):

$$\min_{\mathbf{c}} \|\mathbf{c}\|_1 \ \text{s.t.}\ \mathbf{x}_j = X\mathbf{c},\ c_j = 0,\ \mathbf{1}^T\mathbf{c} = 1 \quad (3)$$

dense !



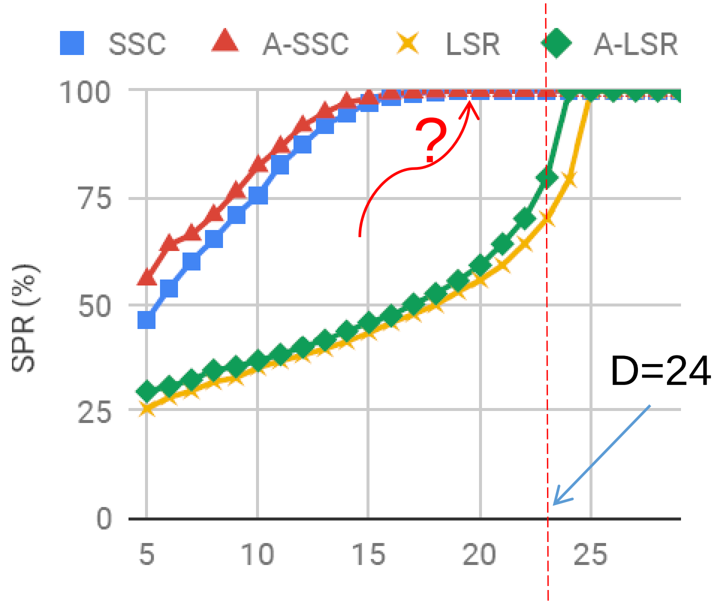
(a) data points (b) coefficients matrix C (c) affinity matrix A

➢ Theorem 1: The affine subspaces $\{\mathcal{A}_\ell\}_{\ell=1}^n$: affinely independent, any optimal solution to (3) is subspace-preserving.

- Affinely independent requires:

$$\dim(\mathrm{aff}(\cup_{\ell=1}^n \mathcal{A}_\ell)) + 1 = \sum_{\ell=1}^n \dim(\mathcal{A}_\ell) + n$$

➢ **Synthetic Data**: n=5 , dim=4



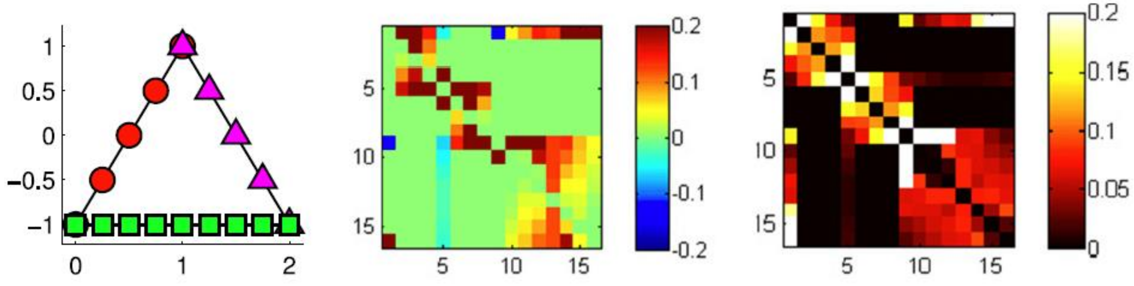
(a) SPR vs. D

[1] C.-G. Li, C. You, and R. Vidal, "On Geometric Analysis of Affine Sparse Subspace Clustering", IEEE J. STSP, Vol.12, Issue 6, pp.1520-1533, Dec. 2018.

On Geometric Analysis of Affine SSC (2/3)

- Affine SSC (ASSC):

$$\min_{\mathbf{c}} \|\mathbf{c}\|_1 \ \text{s.t.}\ \mathbf{x}_j = X\mathbf{c},\ c_j = 0,\ \mathbf{1}^T\mathbf{c} = 1 \quad (3)$$



(a) data points (b) coefficients matrix C (c) affinity matrix A

➢ Relative position of data points:

- In relative interior: $\mathbf{c} \geq 0 \Rightarrow \|\mathbf{c}\|_1 = \mathbf{1}^T\mathbf{c} = 1$
  - ✓ Every optimal solution is *nonnegative*.
  - ✓ Equivalent condition to guarantee subspace-preserving: $\mathcal{A}_\ell$ does not intersect $\mathrm{conv}(\mathcal{X}^{(-\ell)})$
  - ✓ Exists nonnegative subspace-dense solution
- On boundary:
  - ✓ Every optimal solution is nonnegative
- At vertex:
  - ✓ Nonnegative solution cannot be subspace preserving

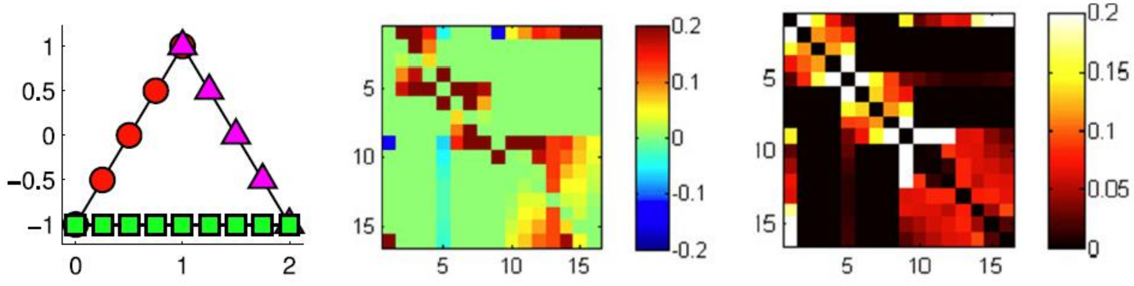
approximately $\sim \dfrac{c(\log N_\ell)^{\frac{d_\ell - 1}{2}}}{N_\ell}$

[1] C.-G. Li, C. You, and R. Vidal, "On Geometric Analysis of Affine Sparse Subspace Clustering", IEEE J. STSP, Vol.12, Issue 6, pp.1520-1533, Dec. 2018.

On Geometric Analysis of Affine SSC (3/3)

- Affine SSC (ASSC):

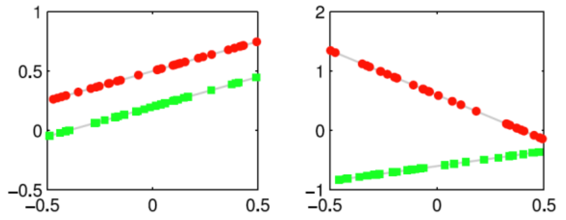
$$\min_{\mathbf{c}} \|\mathbf{c}\|_1 \ \text{s.t.}\ \mathbf{x}_j = X\mathbf{c},\ c_j = 0,\ \mathbf{1}^T\mathbf{c} = 1 \quad (3)$$



(a) data points (b) coefficients matrix C (c) affinity matrix A

➢ Relative position of data points:

- In relative interior: $\mathbf{c} \geq 0 \rightarrow \|\mathbf{c}\|_1 = \mathbf{1}^T\mathbf{c} = 1$
  - ✓ Every optimal solution is *nonnegative*.
  - ✓ Equivalent condition to guarantee subspace-preserving
  - ✓ Exists nonnegative subspace-dense solution
- On boundary:

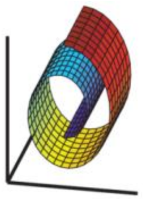


(a) (b)

	Hopkins 155		Iris	Wdbc	Wine	Ionosphere	UCI Digits	USPS
	$n=2$	$n=3$	$n=3$	$n=2$	$n=3$	$n=2$	$n=10$	$n=10$
SSC(n)	5.48	15.60	8.00	28.12	34.83	35.33	31.87	14.71
ASSC(n)	**1.95**	**4.94**	**6.67**	**8.26**	**7.87**	**29.91**	**28.67**	**11.70***

[1] C.-G. Li, C. You, and R. Vidal, “On Geometric Analysis of Affine Sparse Subspace Clustering”, IEEE J. STSP, Vol.12, Issue 6, pp.1520-1533, Dec. 2018.

- Q & A?



(Science,2000)

Thank you!