550.310 Probability and Statistics for the  
Physical and Information Sciences and  
Engineering, Spring 2005  
Exam #3

Monday May 16, 2005

Name:
Section:

Part I: Multiple choice. Clearly circle the letter of the most correct answer  
to each question. Each question is worth 4 points.

1. (4 points) If \( X \) and \( Y \) are independent then \( E(XY) = E(X)E(Y) \)
   
   (a) true
   (b) false

2. (4 points) If \( E(XY) = E(X)E(Y) \) then \( X \) and \( Y \) are independent.
   
   (a) true
   (b) false

3. (4 points) If \( X_1, \ldots, X_n \) are \( n \) Normal random variables then their sum  
is also Normal.
   
   (a) true
   (b) false

4. (4 points) If \( X_1, \ldots, X_n \) are \( n \) Uniform random variables then their sum  
is also Uniform
   
   (a) true
   (b) false
Part II: Problem Solving. You must show all your work to receive credits, or to receive partial credits for incorrect answers. When using the tables, choose the value closest to the exact answer.

1. Let $X$ and $Y$ be continuous random variables with joint probability density function

$$f(x, y) = \begin{cases} x + y, & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) (10 points) Show that the marginal probability density function of $X$ is

$$f(x) = \begin{cases} x + \frac{1}{2}, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(b) (5 points) Are $X$ and $Y$ independent? Justify your answer.

(c) (10 points) Compute $P(X + Y \leq 1)$
2. Suppose that the number of phone calls received by a hotline during any given minute is a Poisson random variable. Its parameter, \( \lambda \), is unknown and must be estimated. In order to do so, the numbers of calls are recorded during 10 minutes. \( x_1, \ldots, x_{10} \) are the numbers of calls during each minute.

(a) (10 points) Show that the maximum likelihood estimator for \( \lambda \) is

\[
\bar{x} = \frac{1}{10} \sum_{i=1}^{10} x_i
\]

(b) (10 points) The actual observations are 2, 0, 1, 2, 3, 3, 2, 0, 3, 4. How would you estimate, using the Poisson hypothesis, the probability to observe exactly one phone call during a given minute.
3. In a certain country, 2 million people are going to elect their president by choosing between candidate A and candidate B. A sample of \( n \) people is chosen at random and the proportion of those that are going to vote for A is evaluated. This proportion will be denoted \( \hat{p} \).

(a) (10 points) Given that we want to be 99% sure that \( \hat{p} \) is within 0.03 of the true proportion, how large a sample do we need? You can use the fact that for all \( 0 \leq x \leq 1 \),
\[
x(1-x) \leq 0.25.
\]

(b) (10 points) Suppose that the actual number of people that will vote for A is 1,020,000. How large a sample would we need to have a 99% chance that \( \hat{p} \) would be over 50%.
4. Let $X$ and $Y$ be 2 independent continuous random variables uniformly distributed over the interval $[0, 1]$. Let $Z = \max(X, Y)$.

(a) (10 points) Compute the pdf of $Z$. You can use the fact that for any $t$,

$$\{Z \leq t\} = \{X \leq t \text{ and } Y \leq t\}$$

for computing the cumulative distribution function of $Z$ and then compute the pdf of $Z$.

(b) (10 points) Compute the pdf of $T = \min(X, Y)$
FORMULA PAGE

X follows a Binomial distribution:
its pmf is
\[
p(x; n, p) = \binom{n}{x}p^x(1-p)^n-x, \quad x = 0, \ldots, n
\]
\[
e = 0 \quad \text{otherwise}
\]

\[E(X) = np, V(X) = np(1-p)\]

X follows a Uniform distribution over the interval \([0, 1] \):
its pmf is
\[
f(x) = 1 \text{ if } 0 \leq x \leq 1
\]
\[
e = 0 \quad \text{otherwise}
\]

\[E(X) = \frac{1}{2}, V(X) = \frac{1}{12}\]

X follows a Poisson distribution:
its pmf is
\[
p(x, \lambda) = \frac{e^{-\lambda x}}{x!}, \quad x = 0, 1, 2, \ldots
\]
\[
e = 0 \quad \text{otherwise}
\]

\[E(X) = V(X) = \lambda\]

X follows a Normal distribution:
its pdf is
\[
f(x, \mu, \sigma) = \frac{1}{\sigma \sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}
\]
\[E(X) = \mu, V(X) = \sigma^2\]

X follows an Exponential distribution:
its pdf is
\[
f(x, \lambda) = \lambda e^{-\lambda x}, \quad x \geq 0
\]
\[
e = 0 \quad x < 0
\]

\[E(X) = \frac{1}{\lambda}, V(X) = \frac{1}{\lambda^2}\]