AN INFINITE DIMENSIONAL GROUP APPROACH FOR PHYSICS BASED MODELS IN PATTERNS RECOGNITION

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ABSTRACT. Non rigid deformations of patterns can be interpreted as the action of an infinite dimensional group $\mathcal{A}$ on a given set $\mathcal{P}$ of patterns. Following Lie group ideas, a small deformation can be well described by an element y of the tangent space at identity $T_e\mathcal{A}$. Given a metric on $T_e\mathcal{A}$, which brings the cost of a small deformation, we show that we can define on $\mathcal{A}$ a left invariant distance $d_{\mathcal{A}}$ which gives the distance between two arbitrary large deformations. We are concerned with various topological and geometrical properties of $\mathcal{A}$. We reformulate in a unified framework many pattern recognition tasks as non linear variationnal problems on $\mathcal{A}$. We show the existence of solutions to these problems and finally, we propose a sub-optimal algorithm to solve three important classes of pattern recognition problems through a gradient algorithm on $\mathcal{A}$ whose convergence is rigorously established.

CONTENTS

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1. Introduction

During the last decade, the use of deformable models in pattern recognition and in pattern matching have became more and more usual. However, the theoretical framework of this approach is not yet fixed and much of the mathematical work is still to be done. At a formal level, the problem can be formulated as follows. Assume that you have a set of "patterns" $\mathcal{P}$ and a set of "actions" or "deformations" or also "transformations" $\mathcal{A}$ such that for each a in $\mathcal{A}$ and each $f \in \mathcal{P}$ we can define the deformation of f by a as a new element of $\mathcal{P}$ denoted af. A natural assumption at this level is that $\mathcal{A}$ has a group structure so that we can define the product of two elements a and $a' \in \mathcal{A}$ denoted aa' and the inverse denoted a^{-1} of an element $a \in \mathcal{A}$. We should also assume quite naturally that $\mathcal{A}$ is acting on $\mathcal{P}$ i.e. $a(a'f) = (aa')f$ and $ef = f$ where e is the identity element of $\mathcal{A}$. Such a situation is quite common, and as a first example, we can choose for $\mathcal{P}$ the subsets of $\mathbb{R}^2$ (binary shapes) and for $\mathcal{A}$ the group of the isometries of $\mathbb{R}^2$ or the group of the affine transformations. Actually, one needs often a larger group allowing non rigid local transformations, to cover for instance the huge variability of biological shapes [7].

An important issue is to define an appropriate distance $d_{\mathcal{A}}$ on $\mathcal{A}$ inducing a distance $d_{\mathcal{P}}$ between the elements of $\mathcal{P}$ by

$$d_{\mathcal{P}}(f_1, f_2) = \inf\{ \ d_{\mathcal{A}}(e, a) \mid af_1 = f_2 \ \}. \tag{1}$$

This approach is quite natural since the distance between two patterns should be a measure of the amount of "deformation" needed to go from f_1 to f_2. However, the symmetry property of the distance $d_{\mathcal{P}}$ ($d_{\mathcal{P}}(f_1, f_2) = d_{\mathcal{P}}(f_2, f_1)$) is fulfilled if $d_{\mathcal{A}}(e, a) = d_{\mathcal{A}}(e, a^{-1})$ so that a property of left invariance $d_{\mathcal{A}}(aa_1, aa_2) = d_{\mathcal{A}}(a_1, a_2)$) or right invariance ($d_{\mathcal{A}}(a_1a, a_2a) = d_{\mathcal{A}}(a_1, a_2)$) of the distance $d_{\mathcal{A}}$ is natural and attractive. Moreover, if $d_{\mathcal{A}}$ is left invariant, then $(a, b) \to d_{\mathcal{A}}(a^{-1}, b^{-1})$ is right invariant so that we can focus on left invariant distances. If $\mathcal{A}$ is a finite dimensional Lie group of transformations, then, choosing a metric on the Lie algebra $T_e\mathcal{A}$ and extending this metric on $\mathcal{A}$ by left multiplication, we can define the associated geodesic distance which is left invariant. In the framework of deformable models, this choice of a metric on $T_e\mathcal{A}$ corresponds to our a priori on the cost of the small deformations and should be designed according to our precise application. This approach, applied on infinite dimensional groups of transformations, has been suggested by R. Azencott in [4] for shapes recognition.

However, for infinite dimensional group of transformations, two important difficulties arise. The first one is to define a Lie group structure on an infinite dimensional group of transformations and the second one is to equip this Lie group with a left invariant metric for which we can define the associated geodesic distance [12, 16]. There is in fact a third difficulty which is that we should keep in mind that we want at the very end to have an appropriate numerical scheme to solve various pattern recognition problems in this framework.

In this paper, we will assume that $\mathcal{P}$ is the set of the measurable functions from a finite dimensional Riemannian compact manifold M without boundary to a finite dimensional manifold X. This definition matches numerous examples. For instance, the closed curves in $\mathbb{R}^2$ correspond to the case $M = \mathbb{R}/\mathbb{Z}$ (1 dimensional torus) and $X = \mathbb{R}^2$ (see [19, 17]). The case of the periodic grey-level images correspond to $M = \mathbb{R}^2/\mathbb{Z}^2$ (2 dimensional torus) and $X = \mathbb{R}$. In both last case, X is a vector space. However, more general situations arise if we work with images with bounded grey-level values ($X = \mathbb{R}^+$ or $X = [a, b]$) or with images where $f(m)$ is an unitary vector in $\mathbb{R}^3$ i.e. $X = S^2$ representing for instance the direction in m of some physical field. The fact that M is without boundary is a bit restrictive. However, this restriction, only technical, could certainly be relaxed and simplifies the results we will prove in this paper.

As group of deformations $\mathcal{A}$, a natural choice is to consider the group $\mathrm{Aut}(M)$ of the homeomorphisms on M with the action $\phi f = f \circ \phi$ for $f \in \mathcal{P}$ and $\phi \in \mathrm{Aut}(M)$ (note that one should consider on $\mathrm{Aut}(M)$ the product $\phi\phi' = \phi' \circ \phi$). However, this choice may be too restrictive in some situations since the transformation does not affect the range of f. For instance, in the case of closed curves in $\mathbb{R}^2$, $f \circ \phi$ is only a change of variable which does not affect the geometric shape of the curve. Hence, in order to modify the range of f, assume that there exists a finite dimensional Lie group G acting on X by the action $(g, x) \to gx$. In the case of closed curves, one can choose $G = \mathbb{R}^2$ with the action $(g, x) \to g + x$. In the case of positive images $X = \mathbb{R}^+$, we can choose as G the multiplicative group $\mathbb{R}^+_*$ where gx denotes here the multiplication in $\mathbb{R}$. The case $X = S^2$ of images of unitary vector in $\mathbb{R}^3$ is more interesting since we can choose for G various groups of matrices acting on S^2. Now, if $C(M, G)$ is the set of the continuous mapping from M to G on which we have the group product given by the pointwise multiplication in G (i.e. $hh'(m) = h(m)h'(m)$), $C(M, G)$ acts on $\mathcal{P}$ through the action $(h, f) \to hf$ where hf is defined by

$$hf(m) = h(m)f(m)\ ;\ m \in M.$$

Now, putting together both previous actions, we will consider $\mathcal{A}_0 = \text{Aut}(M) \ltimes C(M,G)$, and for any $a = (\phi, \theta) \in \mathcal{A}_0$ and $f \in \mathcal{P}$, the action of a on f will be defined by

$$af = \theta(f \circ \phi),$$

where on $\mathcal{A}_0$ we used the semi-direct product given by

$$aa' = (\phi' \circ \phi, \theta(\theta' \circ \phi)),$$

with $a = (\phi, \theta)$ and $a' = (\phi', \theta')$. At this point, we see that $\mathcal{A}_0$ is not a Lie group, even infinite dimensional. We could have chosen for $\mathcal{A}$ the space

$$\mathcal{A}_\infty = \text{Diff}^\infty(M) \ltimes C^\infty(M,G)$$

where $\text{Diff}^\infty(M)$ is the set of smooth diffeomorphisms on M and $C^\infty(M,G)$ the set of smooth mappings from M to G so that we could have seen $\mathcal{A}_\infty$ as an ILB (inverse limit of Banach space) Lie group as defined in [14] or also a smooth Lie group in the sense of Fröhlicher-Kreigl as in [9, 11] with Lie algebra

$$\mathfrak{A}_\infty = \mathfrak{X}(M) \times C^\infty(M, \mathfrak{G}),$$

where $\mathfrak{X}(M)$ is the set of the smooth vector fields on M and $C^\infty(M,\mathfrak{G})$ the set of the smooth mappings from M to the Lie algebra $\mathfrak{G}$ of G. However, we do not want to restrict ourself to smooth transformations and one of the main point of this article will be to look for group of deformations living between $\mathcal{A}_\infty$ and $\mathcal{A}_0$. The principle of our construction will be the following: start from a norm $|\;|_e$ on the Lie algebra $\mathfrak{A}_\infty (= \mathfrak{X}(M) \times C^\infty(M,\mathfrak{G}))$ of $\mathcal{A}_\infty$. Then for any smooth path $t \to Y_t$ from $[0,1]$ to $\mathfrak{A}_\infty$, define the integrated path $t \to a_t$ in $\mathcal{A}_\infty$ as the solution of the equation formally expressed as

$$\frac{da}{dt} = a_t Y_t \tag{2}$$

where ay denotes the left translation of $y \in \mathfrak{A}_\infty$ by the formal differential of the left multiplication by a. Now, considering paths with finite length according to $|\;|_e$ i.e. $\int_0^1 |Y_s|_e ds < \infty$, we will define the sub-group of transformations $\mathcal{A}_B$ as the ending points of all the integrated paths of finite length. Hence, formally,

$$\mathcal{A}_B = \inf\{\ a_1 \mid \int_0^1 |a_s^{-1} \frac{da}{ds}|_e ds < \infty,\ a_0 = e\ \}. \tag{3}$$

One should say here that the previous point of view is nothing but a generalization to infinite dimensional Lie algebra of the construction proposed by R. Palais in [15] for the construction of finite dimensional Lie group of transformations from a finite dimensional Lie sub-algebra $\mathfrak{h}$ of $\mathfrak{X}(M) \times C^\infty(M,\mathfrak{G})$. Here, the condition $a_t^{-1}\frac{da}{dt} \in \mathfrak{h}$ is replaced by the condition of finite length with respect to $|\;|_e$. In our case, the sub-group is parameterized by the choice of the norm $|\;|_e$ and the final sub-group is infinite dimensional.

Then, we will define on $\mathcal{A}_B$ the associated geodesic distance

$$d_B(e,a) = \inf\{ \int_0^1 |a_s^{-1}\frac{da}{ds}|_e ds \mid a_0 = e,\ a_1 = a \}. \tag{4}$$

However, at that step, the group $\mathcal{A}_B$ is nothing else under weak assumption on the norm $|\ |_e$ than the connected component of the identity $\mathcal{A}_\infty^e$ of $\mathcal{A}_\infty$. However, this group is not complete (as a topological space) according to the metric d_B. This lack of completeness could be a serious drawback in an analysis point of view if we want to use some fixed point theorem or to get existence results for ordinary differential equations on $\mathcal{A}_B$ or also for solution of variational problems arising naturally in pattern recognition. Hence, in fact, $\mathcal{A}_B$ should be formally the completion (as a group and not as a vector space!) of $\mathcal{A}_\infty^e$. Note that the completion of a group is not well defined in a general setting. In fact, we will define the element of $\mathcal{A}_B$, as the values at time 1 of flow of time dependent vector fields living in the Banach space $(B, |\ |_e)$ obtained by completion (this time, the usual completion of vector space) of the Lie algebra $\mathfrak{A}_\infty$ according to the norm $|\ |_e$. The counterpart will be that $\mathcal{A}_B$ will not have a smooth Lie group structure, since for instance, its formal tangent space at identity B is no more a Lie algebra.

At this point, we would like to discuss briefly the beautiful work of D. G. Ebin and J. Mardsen on the application of group of diffeomorphisms in the study of the motion of an incompressible fluid [8]. They introduced the ILH Lie group $\mathcal{D}$ of the smooth diffeomorphisms on a compact manifold M and the essence of their method, following the lines of the work of V. Arnold in [2], is to "transfer the problem from the classical equation to a problem of finding geodesics on the group of volume preserving diffeomorphisms, to which the methods of global analysis and infinite dimensional geometry can be applied". They introduced the group $\mathcal{D}^s$ of the diffeomorphisms whose derivative up to the order s are square integrable (in charts). They proved that $\mathcal{D}^s$ has a *strong* structure of infinite dimensional manifold modeled on an Hilbert space, and they consider on $\mathcal{D}^s$ a *weak* left invariant Riemannian structure given at identity by the integral of the pointwise scalar product (with respect to the metric on M) of any two vector fields on M whose derivative up to the order s are square integrable. Our approach, will be in a sense a dual approach, since in our case the left invariant metric could vary greatly from an application to another one, so that the metric will be in fact a parameterization of $\mathcal{A}_B$ and will be in a sense a *strong* Riemannian metric on $\mathcal{A}_B$. However, as a counterpart, we will have only a *weak* differentiable structure on $\mathcal{A}_B$.

In part 2, we precise the conditions on which we could give a rigorous meaning to (2) and we deduce conditions on the norm $|\ |_e$ such that paths of finite length for $|\ |_e$ in the tangent space at identity can be integrated. Then, we give a precise definition of $\mathcal{A}_B$ and of the distance of d_B and we prove in theorem 2.12 that $(\mathcal{A}_B, d_B)$ is a complete metric space. Then, we show that even if $\mathcal{A}_B$ has not a Lie group structure nor a differentiable structure in the usual sense, $\mathcal{A}_B$ can be equipped with a weak differentiable structure modeled over a Banach space, but strong enough to define the exponential mapping and a useful notion of differentiable real valued functions. Moreover, we show in theorem 2.19 that there exists integral paths in $\mathcal{A}_B$ for bounded and strongly Lipschitz vector fields (as defined in definition 2.18) on $\mathcal{A}_B$.

In part 3, we restrict ourself to the important case when $|\ |_e$ is an Hilbertian norm. In this case, the sub-group $\mathcal{A}_B$ has a weak differentiable structure modeled over an Hilbert space. On such Hilbert sub-group, we prove in theorem 3.7 that there exist continuous geodesic curves between any two points in $\mathcal{A}_B$. Moreover, we show in theorem 3.8 that a wide class of variational problems useful in pattern recognition (see below) admit a solution . This two results come from the weak compactness of the strong ball in $L^2([0,1],B)$, the space of square integrable time dependent vector fields and from the continuity of the flow mapping under the weak topology as proved in theorem 3.2. Finally, using again the fact that the tangent spaces are Hilbert spaces, we show that given a differentiable real valued function E, one can define the gradient ∇E and under some additional conditions on E, for every initial deformation a_0, the solution of the formal gradient evolution equation

$$\frac{da}{dt} = -\nabla_{a_t} E. \tag{5}$$

In part 4, we apply this result to three important problems of pattern recognition:

Template fitting: Assume that we single out a pattern $f \in \mathcal{P}$ called the template pattern. Let $L : X \to \mathbb{R}^+$ be a non negative function called the penalty function. The problem of template fitting for the penalty function L is to find $a \in \mathcal{A}_B$ minimizing

$$\int_M L(af)d\mu + \frac{1}{2} d_B(e,a)^2 \tag{6}$$

where e denotes the identity element in $\mathcal{A}$, d_B the left invariant distance on $\mathcal{A}_B$ and μ is the normalized Riemannian measure on M. If $\hat{a}$ is a solution of the problem (we do not discuss for the moment the existence of such a solution), then $\hat{f} = \hat{a}f$ will be called the fit of f according to L.

Patterns classification: Assume here that we single out $f_1, \cdots, f_p$, p patterns in $\mathcal{P}$ as template patterns. Now consider a new pattern $\widetilde{f} \in \mathcal{P}$. We define the similarity of $\widetilde{f}$ with f_i by

$$S(\widetilde{f}, f_i) = \inf_{a \in \mathcal{A}_B} \int_M L(\widetilde{f}(m), (af)(m)) d\mu + \frac{1}{2} d_B(e,a)^2. \tag{7}$$

The value of $L(x, x')$ is usually a kind of distance between x and x' which controls the similarity between points of X. Now, considering the values of $S(\widetilde{f}, f_i)$ for all $i \in \{1, \cdots, p\}$, we can classify $\widetilde{f}$ into one of the classes defined by the f_i's.

Pattern matching: Keeping the notation introduced for the classification problem, we denote $\hat{a}_i$ the element of $\mathcal{A}_B$ (if it exists) achieving the minimal value of $S(\widetilde{f}, f_i)$ for i minimizing the values of the $S(\widetilde{f}, f_j)$'s. Indeed, if $\hat{\phi}_i$ is the homeomorphism corresponding to $\hat{a}_i$, $\hat{\phi}_i$ is a mapping from the points $(m, \widetilde{f}(m))$ of the new pattern to the points $(\hat{\phi}_i(m), f \circ \hat{\phi}_i(m))$ of the template f_i.

We will propose in this last part a sub-optimal solution to these three problems based on a gradient algorithm in $\mathcal{A}_B$ for the function $E(a) = \int_M L(af)d\mu$ in the case of template fitting and $E(a) = \int_M L(\widetilde{f}, af)d\mu$ in the case of pattern classification and pattern matching. This sub-optimal solution can be achieved numerically in various situations as will be shown in a forthcoming paper [18].

2. The abstract construction of $\mathcal{A}_B$

Let us recall briefly the framework and the notations. Let M and X be two finite dimensional manifolds. We assume that M is compact, without boundary, connected and Riemannian. We denote $\mathcal{P}$ the set of all the measurable functions from M to X. The set $\mathcal{P}$ will be called the space of patterns. Now, let G be a finite dimensional connected Lie group and $\mathfrak{G}$ be its Lie algebra. We assume that G acts on X and the action of $g \in G$ on $x \in X$ will be denoted gx. Now, let $C(M,G)$ be the set of all the continuous functions from M to G and $\mathrm{Aut}(M)$ be the set of all the homeomorphisms on M (we will use on $\mathrm{Aut}(M)$ the product $\phi\phi' = (\phi' \circ \phi)$). One can define on $C(M,G)$ a group product $(\theta, \theta') \to \theta\theta'$ for all θ and θ' in $C(M,G)$ where $\theta'' = \theta\theta'$ is defined by $\theta''(m) = \theta(m)\theta'(m)$ and gg' denotes the product on G. Moreover, $C(M,G)$ acts on $\mathcal{P}$ through the following action

$$(\theta f)(m) = \theta(m)f(m) \quad ; \; m \in M, \; \theta \in C(M,G), \; f \in \mathcal{P}.$$

Consider the set $\mathcal{A}_0 = \mathrm{Aut}(M) \ltimes C(M,G)$ which will be called the set of the actions. For each element $a \in \mathcal{A}_0$ we will denote ϕ its component on $\mathrm{Aut}(M)$ and θ its component on $C(M,G)$. As usual, the semi-direct product $(a, a') \to aa'$ on $\mathcal{A}_0$ for all $a = (\phi, \theta)$ and $a' = (\phi', \theta')$ is defined by

$$aa' \quad = \quad (\phi' \circ \phi, \theta(\theta' \circ \phi))$$

where $\circ$ denotes the composition of functions. One verifies easily that for this product, $\mathcal{A}_0$ is a group acting on $\mathcal{P}$ through the following action

$$af = \theta(f \circ \phi),$$

where $a = (\phi, \theta)$.

As suggested in the introduction, we will consider also the subgroup $\mathcal{A}_\infty = \mathrm{Diff}^\infty(M) \ltimes \mathcal{C}^\infty(M,G)$ of $\mathcal{A}_0$ which has a structure of smooth infinite dimensional Lie group (see [11]) whose Lie algebra is given by $\mathfrak{A}_\infty = \mathfrak{X}(M) \times C^\infty(M, \mathfrak{G})$. Let e be the identity element of $\mathcal{A}_\infty$. This element is defined by $e(m) = (m, 1_G)$ for all

$m \in M$ where 1_G denotes the identity element in G. Throughout this work, y will usually denote an element of $\mathfrak{A}_\infty = T_e\mathcal{A}_\infty$ and u (resp. z) its component on $\mathfrak{X}(M)$, (resp. its component on $C^\infty(M, \mathfrak{G})$). For any $a \in \mathcal{A}_\infty$, the left multiplication L_a on $\mathcal{A}_\infty$ given by $L_a(a') = aa'$ is a smooth mapping (see [11]) whose differential at identity is given by

$$d_e(L_a)(y)(m) = \big(u \circ \phi(m), d_{1_G}(L_{\theta(m)})((z \circ \phi)(m))\big) ,$$

where $a = (\phi, h)$ and $d_{1_G}(L_g)$ denotes the usual differential at 1_G of the left multiplication by $g \in G$ on G. In order the simplify the formulation, we will use the notation ay to denote $d_e(L_a)(y)$ and $\theta(z \circ \phi)$ to denote $m \to d_{1_G}(L_{\theta(m)})((z \circ \phi)(m))$ so that we get the new definition

$$ay = (u \circ \phi, \theta(z \circ \phi)). \tag{8}$$

Hence, the tangent space $T_a\mathcal{A}_\infty$ at any point $a \in \mathcal{A}_\infty$ can be expressed by

$$T_a\mathcal{A}_\infty = \{ \ ay \mid y \in \mathfrak{A}_\infty \ \}.$$

Throughout this work, for all $Y \in C^\infty([0,1] \times M, TM \times \mathfrak{G})$, we will denote U its component on TM and Z its component on $\mathfrak{G}$ so that $U \in C^\infty([0,1] \times M, TM)$ and $Z \in C^\infty([0,1] \times M, \mathfrak{G})$. As usual, for $t \in [0,1]$, Y_t denotes the function in $C^\infty(M, TM \times \mathfrak{G})$ defined by $Y_t(m) = Y(t, m)$ for any $m \in M$.

Definition 2.1. Let $\mathcal{T}^\infty$ be defined by

$$\mathcal{T}^\infty = \{ \ Y \in C^\infty([0,1] \times M, TM \times \mathfrak{G}) \mid Y_t \in \mathfrak{A}_\infty \forall t \in [0,1]\}.$$

Let $C([0,1] \times M, M \times G)$ (resp. $C^\infty([0,1] \times M, M \times G)$) be the set of the continuous (resp. smooth) functions from $[0,1] \times M$ to $M \times G$. For any $A \in C([0,1] \times M, M \times G)$, we denote Φ its component on M and Θ its component on G so that $\Phi \in C([0,1] \times M, M)$ and $\Theta \in C([0,1] \times M, G)$. From the classical theory of O.D.E. on smooth manifolds, we deduce that for all $Y \in \mathcal{T}^\infty$, there exists $A \in C^\infty([0,1] \times M, M \times G)$ such that $A_0 = e$ and

$$\frac{\partial A}{\partial t} = A_t Y_t, \tag{9}$$

that is $\Phi_0 = \mathrm{Id}_M$, $\Theta_0 \equiv 1_G$ and

$$\begin{cases} \frac{\partial \Phi}{\partial t}(t,m) = U(t, \Phi(t,m)) \\ \\ \frac{\partial \Theta}{\partial t}(t,m) = \Theta(t,m) Z(t, \Phi(t,m)), \end{cases} \tag{10}$$

where $Y = (U, Z)$ and $A = (\Phi, \Theta)$. Obviously, for any $t \in [0,1]$, $A_t \in \mathcal{A}_\infty$. At this stage, we can define through (9), the flow mapping $\mathbf{A} : \mathcal{T}^\infty \to C([0,1] \times M, M \times G)$ such that for all $Y \in \mathcal{T}^\infty$, $\mathbf{A}(Y) = A$, where A is the solution of (9). Our approach, as presented in the introduction, will be to extend the flow mapping $\mathbf{A}$ to a separable

Banach space $\mathbb{L}^1_B = L^1([0,1], B)$ of time dependent vector fields where $(B, |\ |_e)$ is the completion of $\mathfrak{A}_\infty$ for the norm $|\ |_e$. Then, the subgroup $\mathcal{A}_B$ will be defined by $\mathcal{A}_B = \{\ \mathbf{A}_1(Y) \mid Y \in \mathbb{L}^1_B\ \}$ where $\mathbf{A}_1(Y)$ is the value at time 1 of the flow. Therefore, we will need first some control lemmas on $\mathbf{A}$ given in the following section. At first reading, the reader is invited to skip the proof of the three above lemmas and to go directly to main consequences given in proposition 2.10.

2.1. Control lemmas on the flow mapping. Let $\langle\ ,\ \rangle^M_m$ denotes the metric at point $m \in M$ and ∇^M denotes the Riemannian connection on M. Let 1_G denotes the identity element in G and let $\langle\ ,\ \rangle^G_{1_G}$ be a scalar product on $\mathfrak{G}$ (people unfamiliar with Riemannian geometry and Lie group theory could usefully refer to [6] and [10]). We extend this scalar product on each tangent space T_gG through left multiplication so that G becomes a Riemannian manifold. We denote ∇^G the Riemannian connection on G.

Definition 2.2. (i) For all $u \in \mathfrak{X}(M)$, we define

$$|u|_\infty = \sup\{\ (\langle u(m),\, u(m)\rangle^M_m)^{1/2} \mid m \in M\ \},$$

$$|\nabla u|_\infty = \sup\{\ |\nabla^M_v u|_\infty \mid v \in \mathfrak{X}(M),\ |v|_\infty = 1\ \}.$$

(ii) For all $z \in C^\infty(M, \mathfrak{G})$, we define

$$|z|_\infty = \sup\{\ (\langle z(m),\, z(m)\rangle^G_{1_G})^{1/2} \mid m \in M\ \},$$

$$|\nabla z|_\infty = \sup\{\ |\nabla^{\mathfrak{G}}_v z|_\infty \mid v \in \mathfrak{X}(M),\ |v|_\infty = 1\ \},$$

where $z' = \nabla^{\mathfrak{G}}_v z$ is defined by $z'(m) = d_m z(v(m))$ and $d_m z$ is the differential of z at $m \in M$.

Definition 2.3. Let d_M and d_G denote the distances associated with the Riemannian structure on M and G. We define the distance d_0 on $\mathcal{A}_0$ by

$$d_0(a, a') = \sup_{m\in M} d_M(\phi(m), \phi'(m)) + \sup_{m\in M} d_M(\phi^{-1}(m), (\phi')^{-1}(m)) + \sup_{m\in M} d_G(\theta(m), \theta'(m)),$$

where $a = (\phi, \theta)$ and $a' = (\phi', \theta')$. From the completeness of M and G, we deduce that d_0 defines on $\mathcal{A}_0$ a metric for which $\mathcal{A}_0$ is complete. Now, we define on the set $C([0,1], \mathcal{A}_0)$ of the continuous functions from $[0,1]$ to $(\mathcal{A}_0, d_0)$ the distance D by

$$D(A, A') = \sup_{s\leq 1} d_0(A_s, A'_s).$$

Here again, the space $C([0,1], \mathcal{A}_0)$ is complete for the distance D.

Let K_G be the constant defined by

$$K_G = \sup\{\ |\nabla^G_w W^G(1_G)|_{1_G}\ |\ w, W \in \mathfrak{G},\ |w|_{1_G} = |W|_{1_G} = 1\ \}.$$

where W^G in the left invariant vector fields on $\mathfrak{X}(G)$ defined by $W \in \mathfrak{G}$.

Lemma 2.4. *Let $Y \in \mathcal{T}^\infty$ and denote $A = (\Phi, \Theta)$ the solution of (9). For all m, m' in M and all $t \in [0,1]$, we have*

$$d_M(\Phi_t(m), \Phi_t(m')) \vee d_M(\Phi_t^{-1}(m), \Phi_t^{-1}(m')) \leq d_M(m, m') e^{\int_0^t |\nabla U_s|_\infty ds}, \tag{11}$$

$$d_G(\Theta_t(m), \Theta_t(m')) \leq d_M(m, m') \big(\int_0^t |\nabla Z_s|_\infty ds \big) e^{\int_0^t |\nabla U_s|_\infty + K_G |Z_s|_\infty ds}. \tag{12}$$

Proof. Let $p \in C^\infty([0,1], M)$ be a smooth path such that $p(0) = m$ and $p(1) = m'$. Let $\widetilde{A} = (\widetilde{\Phi}, \widetilde{\Theta}) \in C^\infty([0,1] \times [0,1], M \times G)$ be defined by

$$\widetilde{A}(t,s) = A(t, p(s)).$$

Using covariant derivatives, we get

$$\frac{D}{dt}\left(\frac{\partial \widetilde{\Phi}}{\partial s}\right) = \frac{D}{ds}\left(\frac{\partial \widetilde{\Phi}}{\partial t}\right) = \frac{D}{ds}\left(U(t, \widetilde{\Phi}(t,s))\right) = \nabla_{\frac{\partial \widetilde{\Phi}}{\partial s}} U(t, \widetilde{\Phi}(t,s)).$$

Hence, if $r = \langle \frac{\partial \widetilde{\Phi}}{\partial s}, \frac{\partial \widetilde{\Phi}}{\partial s} \rangle^M_{\widetilde{\Phi}}$, we get $\frac{\partial r}{\partial t} = 2 \langle \frac{D}{dt}\left(\frac{\partial \widetilde{\Phi}}{\partial s}\right), \frac{\partial \widetilde{\Phi}}{\partial s} \rangle^M_{\widetilde{\Phi}} \leq 2 |\nabla U_t|_\infty r$, so that applying Gronwall's lemma we get

$$|\frac{\partial \widetilde{\Phi}}{\partial s}| \leq |\frac{dp}{ds}| \exp(\int_0^t |\nabla U_u|_\infty du), \tag{13}$$

and

$$d_M(\Phi(t,m), \Phi(t,m')) = d_M(\widetilde{\Phi}(t,0), \widetilde{\Phi}(t,1)) \leq e^{\int_0^t |\nabla U_u|_\infty du} \int_0^1 |\frac{dp}{ds}| ds. \tag{14}$$

We deduce the same inequality for $d_M(\Phi_t^{-1}(m), \Phi_t^{-1}(m'))$ since for a fixed t, Φ_t^{-1} is the value at time t of the time reversed velocity field $\widetilde{U}$ given by $\widetilde{U}_s = -U_{t-s}$ for any $0 \leq s \leq t$.

Considering now $\frac{\partial \widetilde{\Theta}}{\partial s}$, we get using covariant derivatives and the left invariance of the metric on G

$$\frac{D}{dt}\left(\frac{\partial \widetilde{\Theta}}{\partial s}\right) = \frac{D}{ds}\left(\frac{\partial \widetilde{\Theta}}{\partial t}\right) = \widetilde{\Theta}\big(\nabla_{\frac{\partial \widetilde{\Phi}}{\partial s}} Z_t + \nabla^G_{\widetilde{\Theta}_t^{-1} \frac{\partial \widetilde{\Theta}}{\partial s}} Z(t, \widetilde{\Phi}(s,t))^G\big), \tag{15}$$

where for all $W \in \mathfrak{G}$, W^G denotes the left invariant vector fields defined by W. Now, let $r_t = \langle \frac{\partial \widetilde{\Theta}}{\partial s}, \frac{\partial \widetilde{\Theta}}{\partial s} \rangle^G$, we get from (15) $\frac{\partial r}{\partial t} \leq 2 r^{1/2} |\nabla Z_t|_\infty |\frac{\partial \widetilde{\Phi}}{\partial s}| + 2 r K_G |Z_t|_\infty$.

Since $r_0 = 0$, we deduce that $\sqrt{r_t} \leq \int_0^t |\nabla Z_u|_\infty |\frac{\partial \widetilde{\Phi}}{\partial s}| \exp(\int_u^t K_G |Z_{u'}|_\infty du') du$, so that using (13) we get

$$(16) \quad d_G(\Theta(t,m), \Theta(t,m')) \leq \int_0^t |\nabla Z_u|_\infty \exp(\int_0^u |\nabla U_{u'}|_\infty du' + \int_u^t K_G |Z_{u'}|_\infty du') \int_0^1 |\frac{dp}{ds}| ds.$$

Since (14) and (16) are true for an arbitrary smooth path on M from m to m', we get (11) and (12). Thus the proof is complete □

Lemma 2.5. *Let $Y = (U, Z)$ be in $\mathcal{T}^\infty$ and let $A = (\Phi, \Theta)$ be the solution of* (9). *Then $A \in C([0,1], \mathcal{A}_0)$ and for any $0 \leq t \leq t' \leq 1$ we have*

$$d_0(A_t, A_{t'}) \leq 2(\int_t^{t'} |U_s|_\infty ds) e^{\int_0^{t'} |\nabla U_s|_\infty ds} + \int_t^{t'} |Z_s|_\infty ds.$$

Proof. Obviously, for all $t \in [0,1]$, $A_t \in \mathcal{A}_\infty$. Moreover, for all $0 \leq t \leq t' \leq 1$ and all $m \in M$, we get easily

$$(17) \qquad d_M(\Phi_t(m), \Phi_{t'}(m)) \leq \int_t^{t'} |U_s|_\infty ds,$$

$$(18) \qquad d_G(\Theta_t(m), \Theta_{t'}(m)) \leq \int_t^{t'} |Z_s|_\infty ds.$$

Now, using inequality (11) in lemma 2.4 and inequality (17), we get

$$\begin{aligned} d_M(\Phi_t^{-1}(m), \Phi_{t'}^{-1}(m)) &= d_M(\Phi_{t'}^{-1}(\Phi_{t'}(\Phi_t^{-1}(m))), \Phi_{t'}^{-1}(m)) \\ &\leq d_M(\Phi_{t'}(\Phi_t^{-1}(m)), m) e^{\int_0^{t'} |\nabla U_s|_\infty ds} \leq (\int_t^{t'} |U_s|_\infty ds) e^{\int_0^{t'} |\nabla U_s|_\infty ds}. \end{aligned}$$

Considering the supremum of the above inequalities over $m \in M$, we get the result □

Lemma 2.6. *Let $Y = (U, Z)$ and $Y' = (U', Z')$ be in $\mathcal{T}^\infty$ and let $A = (\Phi, \Theta)$ (resp. $A' = (\Phi', \Theta')$) be the solution of* (9). *Then for all $t \in [0,1]$ and all $m \in M$, we have*

$$(19) \qquad d_M(\Phi_t(m), \Phi'_t(m)) \vee d_M(\Phi_t^{-1}(m), (\Phi'_t)^{-1}(m)) \leq K(t),$$

$$(20) \quad d_G(\Theta_t(m), \Theta'_t(m)) \leq \int_0^t [|(Z' - Z)_u|_\infty + (|\nabla Z_u|_\infty \vee |\nabla Z'_u|_\infty)\, K(u)]\, e^{\int_u^t K_G(|Z_{u'}|_\infty \vee |Z'_{u'}|_\infty) du'} du.$$

where

$$K(t) = \int_0^t |(U - U')_u|_\infty e^{\int_u^t |\nabla U_{u'}|_\infty \vee |\nabla U'_{u'}|_\infty du'} du.$$

Proof. Consider $\widetilde{Y} \in C^\infty([0,1]\times[0,1]\times M, TM\times\mathfrak{G})$ defined by $\widetilde{Y}(s,t,m) = Y(t,m) + s(Y'(t,m) - Y(t,m))$. As usual, we denote $\widetilde{U}$ its component on TM and $\widetilde{Z}$ its component on $\mathfrak{G}$. There exists $\widetilde{A}$ on $C^\infty([0,1]\times[0,1]\times M, M\times G)$ such that $\frac{\partial \widetilde{A}}{\partial t} = \widetilde{A}_t \widetilde{Y}_t$. We denote $\widetilde{\Phi}$ its component on M and $\widetilde{\Theta}$ its component on G. Using covariant derivatives, we get

$$\frac{D}{dt}\left(\frac{\partial\widetilde{\Phi}}{\partial s}\right) = (U'-U)_t\circ\widetilde{\Phi} + \left(\nabla_{\frac{\partial\widetilde{\Phi}}{\partial s}} U_t + s\nabla_{\frac{\partial\widetilde{\Phi}}{\partial s}}(U'-U)_t\right)\circ\widetilde{\Phi}. \tag{21}$$

Let $r_t = |\frac{\partial\widetilde{\Phi}_t}{\partial s}|^2$. We deduce from (21) and the equality $\frac{\partial r}{\partial t} = 2\langle\frac{D}{dt}\left(\frac{\partial\widetilde{\Phi}}{\partial s}\right), \frac{\partial\widetilde{\Phi}}{\partial s}\rangle^M_{\widetilde{\Phi}}$ that

$$\frac{\partial r}{\partial t} \leq 2(|\nabla U|_\infty \vee |\nabla U'|_\infty)r + 2|U'-U|_\infty\sqrt{r}.$$

Applying the Gronwall's lemma to $\sqrt{r}$, we get finally

$$|\frac{\partial\widetilde{\Phi}}{\partial s}| \leq \int_0^t |U'-U|_\infty e^{\int_u^t |\nabla U|_\infty\vee|\nabla U'|_\infty du'}du, \tag{22}$$

so that $d_M(\Phi_t(m), \Phi'_t(m)) = d_M(\widetilde{\Phi}(0,t,m), \widetilde{\Phi}(1,t,m)) \leq K(t)$. Using the same argument than in the proof of the lemma 2.4, we deduce the same inequality for $d_M(\Phi_t^{-1}(m), (\Phi'_t)^{-1}(m))$. Thus, (19) is proved.

We turn now to the proof of (20). We have

$$\begin{aligned}\frac{D}{dt}\left(\frac{\partial\widetilde{\Theta}}{\partial s}\right) = \frac{D}{ds}\left(\frac{\partial\widetilde{\Theta}}{\partial t}\right) = \frac{D}{ds}\left(\widetilde{\Theta}_t\widetilde{Z}_t\right) \\ = \widetilde{\Theta}_t\left((Z'-Z)_t\circ\widetilde{\Phi}_t + \nabla_{\frac{\partial\widetilde{\Phi}}{\partial s}}\widetilde{Z}_t + \nabla^G_{\widetilde{\Theta}_t^{-1}\frac{\partial\widetilde{\Theta}}{\partial s}}\widetilde{Z}(s,t,\widetilde{\Phi})^G\right),\end{aligned} \tag{23}$$

where W^G in the left invariant vector fields on $\mathfrak{X}(G)$ defined by $W \in \mathfrak{G}$. Let $r_t = |\frac{\partial\widetilde{\Theta}_t}{\partial s}|^2$. We deduce from (23) that

$$\frac{\partial r}{\partial t} \leq 2\left(|(Z'-Z)_t|_\infty + |\nabla\widetilde{Z}_t|_\infty|\frac{\partial\widetilde{\Phi}}{\partial s}|\right)\sqrt{r} + 2K_G|\widetilde{Z}_t|_\infty r.$$

Hence, using the upper bound (22) of $|\frac{\partial\widetilde{\Phi}}{\partial s}|$ and the fact that $|sZ + (1-s)Z'|_\infty \leq |Z|_\infty \vee |Z'|_\infty$, we get

$$\begin{aligned}&|\frac{\partial\widetilde{\Theta}}{\partial s}(s,t,m)| \leq \\ &\int_0^t \left[|(Z'-Z)_u|_\infty + (|\nabla Z_u|_\infty \vee |\nabla Z'_u|_\infty)\, K(u)\right] e^{\int_u^t K_G(|Z_{u'}|_\infty \vee |Z'_{u'}|_\infty)du'}du.\end{aligned} \tag{24}$$

Finally, since $d_G(\Theta_t(m), \Theta'_t(m)) = d_\Theta(\widetilde{\Theta}(0,t,m), \widetilde{\Theta}(1,t,m))$, integrating over s, we get the last result so that the proof is complete. □

2.2. Definition of $\mathcal{A}_B$.

Definition 2.7. Let $L^0_{\mathrm{Id}_M}(M, TM \times \mathfrak{G})$ be defined by

$$L^0_{\mathrm{Id}_M}(M, TM \times \mathfrak{G}) = \{ \ y : M \to TM \times \mathfrak{G} \mid y \text{ is mesurable }, \ \pi \circ y = Id_M \ \},$$

where π is the canonical projection from $TM \times \mathfrak{G}$ to M.

Definition 2.8. Let $(B, |\ |_e)$ be a Banach subspace of $L^0_{\mathrm{Id}_M}(M, TM \times \mathfrak{G})$. We say that $(B, |\ |_e)$ is admissible if it satisfies the following hypothesis:

H1: The Lie algebra $\mathfrak{A}_\infty$ is a dense subspace of $(B, |\ |_e)$. Moreover, the topology induced by $|\ |_e$ on $\mathfrak{A}_\infty$ is weaker than the usual C^∞ topology.

H2: There exists $K > 0$ such that for all $y = (u, z) \in \mathfrak{A}_\infty$ we have

$$|u|_\infty + |\nabla u|_\infty + |z|_\infty + |\nabla z|_\infty \leq K |y|_e,$$

From now, we consider a fixed admissible Banach space $(B, |\ |_e)$.

Notation 2.9. We will note $\mathbb{L}^0_B = L^0([0,1], B)$ the space of the measurable time dependent vector fields from $[0,1]$ to B. Moreover, for all $p \in [1, +\infty]$, we will note $\mathbb{L}^p_B = L^p([0,1], B)$. The usual norm on $\mathbb{L}^p_B$ will be denoted $||\ ||_p$.

Remark 1. Since the topology induced by $|\ |_e$ is weaker than the C^∞ topology on $\mathfrak{A}_\infty$, we deduce that $(\mathfrak{A}_\infty, |\ |_e)$ is separable and $\mathcal{T}^\infty \subset C([0,1], B) \subset \mathbb{L}^p_B$ for all $p \in [1, \infty]$. Now, since $\mathfrak{A}_\infty$ is dense in $(B, |\ |_e)$, we deduce that$(B, |\ |_e)$ is separable as well as $\mathbb{L}^p_B$ for $p \in [1, \infty[$ and that $\mathcal{T}^\infty$ is dense in $(\mathbb{L}^p_B, ||\ ||_p)$ for all $p \in [1, \infty[$.

Remark 2. Note that the Banach space $(B, |\ |_e)$ can be identified with the completion of $\mathfrak{A}_\infty$ for the norm $|\ |_e$. Moreover, the condition (H1) and (H2) depends only on the behavior of $|\ |_e$ on $\mathfrak{A}_\infty$ so that the condition of admissibility is in fact a condition on the choice of a norm $|\ |_e$ on $\mathfrak{A}_\infty$. The condition on $|\ |_e$ is not restrictive and will be easily checked in most of the particular cases (see for instance the discussion at the end of the paper).

Proposition 2.10. *Let $\mathbf{A} : \mathcal{T}^\infty \to C([0,1], \mathcal{A}_0)$ be such that for all $Y \in \mathcal{T}^\infty$, $\mathbf{A}(Y)$ is the solution of (9). Then,*

(i) there exists $K > 0$ such that for all Y and Y' in $\mathcal{T}^\infty$

$$D(\mathbf{A}(Y), \mathbf{A}(Y')) \leq K ||Y - Y'||_1 e^{K(||Y||_1 + ||Y'||_1)},$$

(ii) the application $\mathbf{A}$ is Lipschitz, uniformly on bounded set, for the norm $||\ ||_1$ on $\mathcal{T}^\infty$ and the distance D on $C([0,1], \mathcal{A}_0)$,

(iii) since $(C([0,1], \mathcal{A}_0), D)$ is a complete metric space, $\mathbf{A}$ has an unique extension on $\mathbb{L}^1_B$.

Proof. The proof is a straightforward consequence of lemma 2.6 □

Definition 2.11. From proposition 2.10, one can define an application $\mathbf{a} : \mathbb{L}^1_B \to \mathcal{A}_0$ by $\mathbf{a}(Y)(m) = \mathbf{A}(Y)(1,m)$ for all $Y \in \mathbb{L}^1_B$ and all $m \in M$.

We can state the main result of this section.

Theorem 2.12. *Let $(B, |\ |_e)$ be an admissible Banach space. Let $\mathcal{A}_B$ be the subset of $\mathcal{A}_0$ defined by $\mathcal{A}_B = \{\ \mathbf{a}(Y) \mid Y \in \mathbb{L}^1_B\ \}$. Then, the set $\mathcal{A}_B$ is a sub-group of $\mathcal{A}_0$. Moreover, if $d_B : \mathcal{A}_B \times \mathcal{A}_B \to \mathbb{R}^+$ is defined by*

$$d_B(a, a') = \begin{cases} \inf\{\ ||Y||_1 \mid Y \in \mathbb{L}^1_B,\ \mathbf{a}(Y) = a'\ \} & \text{if } a = e, \\ \\ d_B(e, a^{-1}a') & \text{otherwise,} \end{cases}$$

where a^{-1} is the inverse of a in $\mathcal{A}$, then the application d_B defines a left invariant distance on $\mathcal{A}_B$ for which $(\mathcal{A}_B, d_B)$ is complete.

Proof. To prove that $\mathcal{A}_B$ is a sub-group of $\mathcal{A}_0$, let us first introduce some notations. For all Y and Y' in $\mathbb{L}^1_B$, we define $Y \star Y' \in \mathbb{L}^1_B$ by

$$(Y \star Y')_t = 2(Y_{2t}\mathbf{1}_{t \le 1/2} + Y'_{2t-1/2}\mathbf{1}_{t > 1/2}).$$

Moreover, for all $Y \in \mathbb{L}^1_B$, we define $S(Y) \in \mathbb{L}^1_B$ by $S(Y)_t = -Y_{1-t}$. Note that $||Y \star Y'||_1 = ||Y||_1 + ||Y'||_1$ and that $||S(Y)||_1 = ||Y||_1$. For all Y and Y' in $\mathbb{L}^1_B$ we have

$$\mathbf{a}(Y \star Y') = \mathbf{a}(Y)\mathbf{a}(Y') \text{ and } \mathbf{a}(Y)\mathbf{a}(S(Y)) = e. \tag{25}$$

Indeed, using a density argument, it is sufficient to prove the result for Y and $Y' \in \mathcal{T}^\infty$. The proof of the first equality is then straightforward. For the second one, one just have to check by derivation that

$$\mathbf{A}(S(Y))_t\mathbf{A}(Y)_1 = \mathbf{A}(Y)_{1-t}\ ;\ t \in [0, 1]. \tag{26}$$

Hence, we get that $\mathcal{A}_B$ is stable for the product and the inverse. Since $e \in \mathcal{A}_B$, we have proved that $\mathcal{A}_B$ is a sub-group of $\mathcal{A}_0$. Let us show that d_B is a distance. From proposition 2.10, we get that $d_0(e, a) \le K d_B(e, a) e^{K d_B(e,a)}$ so that $d_B(e, a) = 0$ implies that $a = e$. Now, from (26), we deduce that for all a, $a' \in \mathcal{A}_B$ and all $Y \in \mathbb{L}^1_B$ such that $a' = a\mathbf{a}(Y)$, we have $a = a'\mathbf{a}(S(Y))$. Since we have $||Y||_1 = ||S(Y)||_1$ we deduce that $d_B(a, a') = d_B(a', a)$ and d_B is symmetric. Finally, let a, a' and a'' be three points in $\mathcal{A}_B$ and let Y and Y' be in $\mathbb{L}^1_B$ such that $a' = a\mathbf{a}(Y)$ and $a'' = a'\mathbf{a}(Y')$. From (25) we get

$$a'' = a\mathbf{a}(Y)\mathbf{a}(Y') = \mathbf{a}(Y \star Y').$$

Since $||Y \star Y'||_1 = ||Y||_1 + ||Y'||_1$, we deduce immediately that d_B satisfies the triangle inequality so that d_B is a distance.

We will prove now that $\mathcal{A}_B$ is complete. Let us first introduce a family of operators on $\mathbb{L}^1_B$. Consider the sequence $(t_k)_{k\in\mathbb{N}}$ defined by $t_k = 1-2^{-k}$. For all p, $q \in \mathbb{N}$ such that $0 \le p < q$, we consider the application $M_{p,q} : (\mathbb{L}^1_B)^{q-p} \to \mathbb{L}^1_B$

$$M_{p,q}(Y_p, \cdots, Y_{q-1}) = \sum_{k=p}^{q-1} 2^{k+1} Y_k(2^{k+1}(t-t_k)) \mathbf{1}_{t_k \le t < t_{k+1}}.$$

One easily verifies that for $p < q < r$ we have the following properties

$$\mathbf{a}(M_{p,q}(Y_p, \cdots, Y_{q-1})) = \mathbf{a}(Y_p) \cdots \mathbf{a}(Y_{q-1}), \tag{27}$$

$$||M_{p,r}(Y_p, \cdots, Y_{r-1} - M_{p,q}(Y_p, \cdots, Y_{q-1})||_1 = \sum_{k=q}^{r-1} ||Y_k||_1. \tag{28}$$

Let now $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{A}_B$. We assume that $\sum_{n\in\mathbb{N}} d_B(a_n, a_{n+1})$ is finite. Thus, there exists a sequence $(Y_n)_{n\in\mathbb{N}}$ in $\mathbb{L}^1_B$ such that $\sum_{n\in\mathbb{N}} ||Y_n||_1 < +\infty$ and $a_{n+1} = a_n \mathbf{a}(Y_n)$. Since the sequence $(a_n)_{n\in\mathbb{N}}$ is bounded in $(\mathcal{A}_B, d_B)$, we deduce from proposition 2.10 (i) that there exists $K' > 0$ such that $d_0(a_n, a_{n+p}) \le K' d_B(a_n, a_{n+p})$. Hence $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{A}_0, d_0)$ so that there exists $a_\infty \in \mathcal{A}_0$ such that $d_0(a_\infty, a_n) \to 0$. From equality (28), the limit in q for fixed p in $\mathbb{L}^1_B$ of $M_{p,q}(Y_p, \cdots, Y_{q-1})$ exists. Let $\widetilde{Y}_p$ be this limit. From equality (27) we deduce that $a_\infty = a_p \mathbf{a}(\widetilde{Y}_p)$. This last equality shows that $a_\infty \in \mathcal{A}_B$ and that $d_B(a_\infty, a_p) \le \sum_{k=p}^{\infty} ||Y_k||_1$. The proof is complete □

Remark 3. Since $\mathbf{a}(Y)$ is invariant under the time change $Y^\lambda_t = \frac{d\lambda}{dt}(t) Y_{\lambda(t)}$ for any smooth strictly increasing time change from $[0,1]$ to $[0,1]$ and any $Y \in \mathbb{L}^1_B$, we deduce easily that for any $p \in [1, +\infty]$

$$\mathcal{A}_B = \{ \mathbf{a}(Y) \mid Y \in \mathbb{L}^p_B \} \text{ and } d_B(e,a) = \inf_{Y \in \mathbb{L}^p_B} ||Y||_p.$$

We end this section by proving that the topology induced by d_0 on $\mathcal{A}_B$ is weaker than the topology induced by d_B.

Proposition 2.13. *There exists $K > 0$ such that for all a and $a' \in \mathcal{A}_B$, we have*

$$d_0(a,a') \le K d_B(a,a') \exp(K(d_B(a,a') + d_B(e,a))).$$

Proof. Let $\widetilde{d}_0$ be the metric on $\mathcal{A}_0$ defined by

$$\widetilde{d}_0(a,a') = \sup_{m\in M} d_M(\phi(m), \phi'(m)) + \sup_{m\in M} d_G(\theta(m), \theta'(m)),$$

for all $a = (\phi, \theta)$ and $a' = (\phi', \theta') \in \mathcal{A}_0$. Then, we deduce from proposition 2.10, that there exists $K > 0$ such that for any $a \in \mathcal{A}_B$ we have $\widetilde{d}_0(e,a) \le K d_B(e,a) \exp(K d_B(e,a))$. Moreover, one easily verifies that $\widetilde{d}_0(a,a') = d_0(e, a^{-1}a')$ (this is not true for d_0) so that $\widetilde{d}_0(a,a') \le K d_B(a,a') \exp(K d_B(a,a'))$. Hence, it is sufficient now to control $\sup_{m\in M}(\phi^{-1}(m), (\phi')^{-1}(m))$ by a right hand term as in the proposition to get the result for any $a = (\phi, \theta)$ and $a' = (\phi', \theta') \in \mathcal{A}_B$. However, using successively lemma 2.4 and lemma 2.5, we get that there exists $K > 0$ such that

$$\begin{aligned} d_M(\phi^{-1}(m), (\phi')^{-1}(m)) &\leq d_M(\phi^{-1}(m), \phi^{-1}(\phi \circ (\phi')^{-1}(m))) \\ &\leq d_m(\phi \circ (\phi')^{-1}(m), m) \exp(K d_B(e, a)) \\ &\leq K d_B(a, a') \exp(K(d_B(a, a') + d_B(e, a))), \end{aligned}$$

so that the result is proved. □

2.3. A weak differentiable structure on $\mathcal{A}_B$.

2.3.1. *Tangent spaces.* The differential of the left multiplication L_a on $\mathcal{A}_\infty$ at identity can be extended through the expression (8) to any $a \in \mathcal{A}_0$ and any $y \in B$ by the same formula

$$ay = (u \circ \phi, \theta(z \circ \phi)); \; a = (\phi, \theta) \in \mathcal{A}_0, \; y = (u, z) \in B.$$

Hence, we will note for any $a \in \mathcal{A}_B$

$$\widetilde{T}_a \mathcal{A}_B = \{ \; ay \mid y \in B \; \}.$$

The vector space $\widetilde{T}_a \mathcal{A}_B$ will be considered as the tangent space of $\mathcal{A}_B$ so that $y \to ay$ will be a mapping from $\widetilde{T}_e \mathcal{A}_B$ to $\widetilde{T}_a \mathcal{A}_B$ transforming Eulerian velocity fields to Lagrangian velocity fields. However, for any arbitrary admissible Banach space B, $\mathcal{A}_B$ cannot be equipped with an usual structure of manifold modeled on a Banach space so that we use the notation $\widetilde{T}_a \mathcal{A}_B$ and avoid the usual notation $T_a \mathcal{A}_B$. Now, we can equip $\mathcal{A}_B$ with a natural left invariant metric if we define on $\widetilde{T}_a \mathcal{A}_B$ the norm $| \; |_a$ such that $y \to ay$ becomes an isometry

$$|ay|_a = |y|_e; \; y \in B.$$

Therefore, $(\widetilde{T}_a \mathcal{A}_B, | \; |_a)$ becomes a separable Banach space for any $a \in \mathcal{A}_B$.

2.3.2. *Exponential.*

Definition 2.14. (i) Let $j : \widetilde{T}_e \mathcal{A}_B \to \mathbb{L}^1_B$ be defined by $j(y)_t = y$.
(ii) Let $\exp : \widetilde{T}_e \mathcal{A}_B \to \mathcal{A}_B$ be defined by $\exp = \mathbf{a} \circ j$.
(iii) For all $a \in \mathcal{A}_B$, we define $\exp_a : \widetilde{T}_a \mathcal{A}_B \to \mathcal{A}_B$ by $\exp_a(y) = a \exp(a^{-1} y)$.

Remark 4. The notation exp comes from the fact that for all $y \in \mathfrak{A}_\infty$, for all $m \in M$, the application $t \to \exp(ty)(m) \in C^\infty(\mathbb{R}, G \times M)$ satisfies

$$\exp(0y) = e \text{ and } \frac{d}{dt}(\exp(ty)(m)) = (\exp(ty)y)(m),$$

so that $t \to \exp(ty)$ is a morphism from $\mathbb{R}$ to $\mathcal{A}_B$. More generally, for all $a \in \mathcal{A}_B$, all $y \in \widetilde{T}_a\mathcal{A}$ and all $m \in M$, the application $t \to \exp_a(ty)(m) \in C^\infty(\mathbb{R}, G \times M)$ satisfies

$$\exp_a(0y) = a \text{ and } \frac{d}{dt}(\exp_a(ty)(m)) = (\exp_a(ty)y)(m),$$

2.3.3. *Differentiable applications.* In spite of the fact that we have only a weak notion of differentiable structure on $\mathcal{A}_B$, we will define the differentiability for functions on $\mathcal{A}_B$

Definition 2.15. Let E be a function from $\mathcal{A}_B$ to $\mathbb{R}$

(i) We say that E is differentiable at $a \in \mathcal{A}_B$, if $E \circ \exp_a$ from the Banach space $\widetilde{T}_a\mathcal{A}_B$ to $\mathbb{R}$ is differentiable at $0 \in \widetilde{T}_a\mathcal{A}_B$ in the usual sense. We will use the notation d_aE to denote $d_0(E \circ \exp)$.

(ii) We say that E is differentiable on $\mathcal{A}_B$ if E is differentiable at any point $a \in \mathcal{A}_B$.

2.3.4. *Important examples.* We are concerned here by a relevant example of differentiable applications in the context of pattern recognition.

Definition 2.16. (i) Let $R = M \times G$ on which we consider the metric defined for all $r = (r_1, r_2) \in R$ by $\langle w, w' \rangle_r^R = \langle w_1, w_1' \rangle_{r_1}^M + \langle w_2, w_2' \rangle_{r_2}^G$ where $w = (w_1, w_2)$ and $w' = (w_1', w_2')$ are elements of $T_rR = T_{r_1}M \times T_{r_2}G$.

(ii) For all $g \in C^2(R, \mathbb{R})$, we define

$$|\nabla^R g|_\infty = \sup\{\ |\nabla^R g(r)| \mid r \in X \times R\ \},$$
$$|\nabla^R\nabla^R g|_\infty = \sup\{\ |\nabla^R_w\nabla^R g(r)| \mid (r,w) \in R \times T_rR,\ |w| \leq 1\ \},$$

where ∇^R is the Riemannian connection on R.

Theorem 2.17. *Assume that the action $(g,x) \to gx$ is C^2 and let f be a C^2 pattern i.e. $f \in C^2(M, X)$.*

(i) Let $L \in C^2(X, \mathbb{R})$ such that $|\nabla^R l|_\infty + |\nabla^R\nabla^R l|_\infty < +\infty$, where $l \in C^2(R, \mathbb{R})$ is defined by $l(r) = L(r_2 f(r_1))$. Then, the function $E : \mathcal{A}_B \to \mathbb{R}$ defined by $E(a) = \int_M L(af)d\mu$ is differentiable on $\mathcal{A}_B$ and for all $a \in \mathcal{A}_B$ and all $y \in \widetilde{T}_a\mathcal{A}_B$

$$d_aE(y) = \int_M \langle \nabla^R l(a),\, y \rangle^R d\mu.$$

(ii) Let $L \in C^2(X \times X, \mathbb{R})$ such that $\sup_{x \in X}(|\nabla^R l_x|_\infty + |\nabla^R\nabla^R l_x|_\infty) < +\infty$. where $l \in C^2(X \times R, \mathbb{R})$ is defined by $l(x,r) = L(x, r_2 f(r_1))$ and l_x by $l_x(r) = l(x,r)$. Let $\widetilde{f} \in \mathcal{P}$ such that $\widetilde{f}(M)$ is relatively compact. Then the

function $E : \mathcal{A}_B \to \mathbb{R}$ *defined by* $E(a) = \int_M L(\widetilde{f}, af)d\mu$ *is differentiable on* $\mathcal{A}_B$ *and for all* $a \in \mathcal{A}_B$ *and* $y \in \widetilde{T}_a\mathcal{A}_B$

$$d_aE(y) = \int_M \langle \nabla^R l(\widetilde{f}, a),\, y\rangle^R d\mu$$

Remark 5. In part (ii), the assumption on the relative compactness of $\widetilde{f}(M)$ is just to ensure that $L(\widetilde{f}, af)$ is integrable.

Proof. Since part (ii) implies obviously part (i), we will proved only the second part.

Let $a \in \mathcal{A}_B$, $y = (u, z) \in \mathfrak{A}_\infty$, $m \in M$ and assume that $|y|_e \le 1$. Now, consider the applications $r = (r_1, r_2) \in C^\infty([0,1], R)$ and $q \in C^2([0,1], \mathbb{R})$ defined by $r(s) = exp_a(sy)(m)$ and $q(s) = l(\widetilde{f}(m), r(s))$. We have $\frac{dq}{ds} = \langle \nabla^R l,\, \frac{dr}{ds}\rangle^R$, and

$$\frac{d^2q}{ds^2} = \langle \nabla^R_{\frac{dr}{ds}} \nabla^R l,\, \frac{dr}{ds}\rangle^R + \langle \nabla^R l,\, \frac{D^R}{ds}\left(\frac{dr}{ds}\right)\rangle^R \tag{29}$$

where $\frac{D^R}{ds}$ is the covariant derivative on R. Since

$$\frac{D^R}{ds}\left(\frac{dr}{ds}\right) = \left(\frac{D^M}{ds}\left(\frac{dr_1}{ds}\right), \frac{D^G}{ds}\left(\frac{dr_2}{ds}\right)\right),$$

$\frac{dr_1}{ds} = u(r_1)$ and $\frac{dr_2}{ds} = r_2z(r_1)$, we get $\frac{D^M}{ds}\left(\frac{dr_1}{ds}\right) = \nabla^M_u u(r_1)$ and $\frac{D^G}{ds}\left(\frac{dr_2}{ds}\right) = r_2(\nabla^{\mathfrak{G}}_{\frac{dr_1}{ds}} z + \nabla^G_{z(r_1)} z(r_1)^G)$, so that $|\frac{D^R}{ds}(\frac{dr}{ds})| \le (|\nabla u|_\infty + +K_G|z|^2_\infty + |\nabla z|_\infty)|u|_\infty \le K|y|^2_a$. Moreover we have $|\frac{dr}{ds}|^2 = |r_1 z(m)|^2 + |u(r_2)|^2 \le |y|^2_a$. Thus we deduce from (29) that $|\frac{d^2q}{ds}| \le M|y|^2_a$, where $M = K\sup_{x\in X}(|\nabla^R l_x|_\infty + |\nabla^R\nabla^R l_x|_\infty)$ and K depends only on M, G and $|\ |_e$. Hence, by integration by parts we get

$$|q(1) - q(0) - \frac{dq}{ds}(0)| \le \int_0^1 (1-s)|\frac{d^2q}{ds}|ds \le M|y|^2_a.$$

Since $\frac{dr}{ds}(0) = y(m)$, $\frac{dq}{ds}(0) = \langle \nabla^R l,\, y(m)\rangle^R$ and

$$|\int_M \langle \nabla^R l,\, y\rangle^R d\mu| \le \sup_{x\in X} |\nabla^R l_x|_\infty |y|_a,$$

we get the result. The proof of the theorem is complete □

2.4. Integration of vector fields on $\mathcal{A}_B$.

Definition 2.18. Let F be a vector fields on $\mathcal{A}_B$ i.e. an application from $\mathcal{A}_B$ to $\widetilde{T}\mathcal{A}_B$ such that $F(a) \in \widetilde{T}_a\mathcal{A}_B$ for all $a \in \mathcal{A}_B$.

(i) We say that F is bounded if there exists $K > 0$ such that

$$\sup_{a\in\mathcal{A}_B} |F(a)|_a \le K.$$

(ii) We say that F is strongly Lipschitz if there exists $K > 0$ such that for all a and a' in $\mathcal{A}_B$ we have

$$|a^{-1}F(a) - (a')^{-1}F(a')|_e \leq Kd_0(a, a').$$

Since $d_0(a, a') \leq Kd_B(a, a')$ on bounded set in $(\mathcal{A}_B, d_B)$ (cf proposition 2.13), if F is strongly Lipschitz, then F is Lipschitz in the usual sense.

Theorem 2.19. *Let F be a bounded and strongly Lipschitz vector field on $\mathcal{A}_B$. Let $\hat{F} : \mathcal{A}_B \to \widetilde{T}_e\mathcal{A}_B$ by defined by $\hat{F}(a) = a^{-1}F(a)$. Then there exists $p \in C([0,1], \mathcal{A}_B)$ such that*

$$p = \mathbf{A}(\hat{F} \circ p). \tag{30}$$

The equation (30) is nothing more than an integrated version of the formal evolution equation

$$\frac{\partial p}{\partial t} = F \circ p.$$

Note that since p is assumed to be in $C([0,1], \mathcal{A}_B)$ and since F Lipschitz, we have $\hat{F} \circ p \in C([0,1], B) \subset \mathbb{L}^1_B$ and equality (30) is well defined.

2.4.1. *Proof of the theorem* 2.19. We will use an iterative scheme. We will built a a sequence of approximates $p_n \in C([0,1], \mathcal{A}_B)$ by induction:

(i) For all $t \in [0,1]$, $p_0(t) = e$,
(ii) $p_{n+1} = \mathbf{A}(Y_n)$ where $Y_n = \hat{F}(p_n)$.

The sequence p_n is not defined until we have proved that $Y_n \in \mathbb{L}^1_B$. However, for $p \in C([0,1], \mathcal{A}_B)$, we have $\hat{F} \circ p \in C([0,1], B) \subset \mathbb{L}^1_B$. Moreover, $A(Y) \in C([0,1], \mathcal{A}_B)$ for all $Y = \hat{F} \circ p$. Indeed, note that if $Y \in C([0,1], B)$, then for all $0 \leq s \leq t \leq 1$ we have $d_B(\mathbf{A}(Y)_s, \mathbf{A}(Y)_t) \leq \int_s^t |Y_u|_e du$. Hence, the sequence $(p_n)_{n\in\mathbb{N}}$ is well defined.

Lemma 2.20. *The sequence $(p_n)_{n\in\mathbb{N}}$ converges in $(C([0,1], \mathcal{A}_0), D)$.*

Proof. Using proposition 2.10 and the fact that F is bounded, we deduce that there exists $K > 0$ such that $d_0(p_{n+1}(t), p_n(t)) \leq K \int_0^t |\hat{F}(p_n) - \hat{F}(p_{n-1})|_e ds$. Hence, if we denote $r_n(t) = \sup_{s\leq t} d_0(p_{n+1}(s), p_n(s))$, we get that there exists K' (independent of n) such that $r_n(t) \leq K' \int_0^t r_{n-1}(s)ds$, so that $r_n(1) \leq \frac{(K')^n}{n!} r_0(1)$, and the proof is complete. □

From the previous lemma, we get that there exists $a_\infty \in C([0,1], \mathcal{A}_0)$ which is the limit of a_n. To prove that a_∞ is the solution of our problem, we have to show that $a_\infty \in C([0,1], \mathcal{A}_B)$. The key argument is the following.

Lemma 2.21. *The sequence $(Y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{L}^1_B$.*

Proof. Since F is strongly Lipschitz, we have $||Y_{n+1} - Y_n||_1 \leq Kr_n(1)$ where r_n is defined in the previous proof. Hence the proof is complete. □

From the previous lemma, we can define Y_∞ as the limit of Y_n in $\mathbb{L}^1_B$ so that $a_\infty = A(Y_\infty)$. Moreover, using again the fact that F is strongly Lipschitz, we get that there exists $K > 0$ such that $||\hat{F} \circ a_\infty - Y_\infty||_1 \le KD(a_n, a_\infty)$ so that we obtain that $Y_\infty = \hat{F} \circ a_\infty$. This complete the proof of the theorem.

2.4.2. *Convergence of the Cauchy approximates.* Let $0 = t_0 \le t_1 \le \cdots \le t_n = 1$ be a subdivision denoted σ of the interval $[0, 1]$. We define the Cauchy approximate associated to σ of a_∞ by

- $a^\sigma(0) = e$,
- $a^\sigma(t) = \exp_{a^\sigma(t_k)}((t - t_k)\hat{F}(a^\sigma(t_k))$ for $t_k \le t < t_{k+1}$.

The path a^σ is obtained through the approximation of $\hat{F}(a^\sigma(t))$ by $\hat{F}(a^\sigma(t_k))$ for $t \in [t_k, t_{k+1}[$. We will show that there exists $K > 0$ such that

$$d_0(a^\sigma, a_\infty) \le K|\sigma|, \tag{31}$$

where $|\sigma| = \sup_{0 \le k < n}(t_{k+1} - t_k)$.

Indeed, let $Y^\sigma = \sum_{k=0}^{n-1} \hat{F}(a^\sigma(t_k))\mathbf{1}_{t_k \le t < t_{k+1}}$. We easily check that $a^\sigma = \mathbf{A}(Y^\sigma)$. Hence, since F is bounded, we get from the proposition 2.10 that there exists $K > 0$ such that

$$\begin{aligned} d_0(a_\infty(t), a^\sigma(t)) &\le K \int_0^t |\hat{F}(a_\infty(s)) - Y^\sigma(s)|_e ds \\ &\le K \int_0^t |\hat{F}(a_\infty(s)) - \hat{F}(a^\sigma(s))|_e ds + K \int_0^t |\hat{F}(a^\sigma(s)) - Y^\sigma(s)|_e ds. \end{aligned}$$

Now, since F is strongly Lipschitz, we get first that there exists $K > 0$ (independent of σ) such that $|\hat{F}(a^\sigma(s)) - Y^\sigma(s)|_e \le K|\sigma|$ and $|\hat{F}(a_\infty(s)) - \hat{F}(a^\sigma(s))|_e \le Kd_0(a_\infty(s), a^\sigma(s))$. Then, using Gronwall's lemma, we obtain the inequality (31).

3. Hilbert sub-groups

In this part, we are concerned by the interesting particular case where the Banach space B is in fact an Hilbert space.

Definition 3.1. Let $(B, |\ |_e)$ be an admissible Banach space. We say that $\mathcal{A}_B$ is a Hilbert sub-group of $\mathcal{A}_0$ if there exists a scalar product $\langle\ ,\ \rangle_e$ on B such that $|y|_e = (\langle y,\, y\rangle_e)^{1/2}$ for all $y \in B$.

Finally, defining the scalar product $\langle\ ,\ \rangle_a$ on $\widetilde{T}_a\mathcal{A}_B$ by

$$\langle y,\, y'\rangle_a = \langle a^{-1}y,\, a^{-1}y'\rangle_e,$$

we get that for all $y \in \widetilde{T}_a\mathcal{A}_B$, $|y|_a = (\langle y,\, y\rangle_a)^{1/2}$.

Hilbert sub-group are of particular interest for several reasons. The main one is that the space $\mathbb{L}^2_B$ becomes an Hilbert space and we can deduce from the weak compactness of the strong unit ball, the compactness for the metric d_0 of the strong balls around e in $\mathcal{A}_B$ for the metric d_B. If fact, we will proved a stronger result since will prove that the flow mapping $\mathbf{A} : \mathbb{L}^2_B \to C([0,1], \mathcal{A}_0)$ is continuous for the weak topology on $\mathbb{L}^2_B$. From this, we will deduce easily the above result of compactness but also the existence of geodesic curves for the Riemannian structure on $\mathcal{A}_B$ and the existence of solution of some variational problems arising from speech recognition.

3.1. Continuity for the weak topology on $\mathbb{L}^2_B$ of the flow mapping. Let us state precisely the result.

Theorem 3.2. *The flow mapping* $\mathbf{A} : \mathbb{L}^2_B \to C([0,1], \mathcal{A}_0)$ *is continuous for the weak topology on* $\mathbb{L}^2_B$ *and the metric* D *on* $C([0,1], \mathcal{A}_0)$.

Proof. The proof is split in two propositions proved in the next section. We will show first in proposition 3.3 that the image of a strong ball in $\mathbb{L}^2_B$ by $\mathbf{A}$ is relatively compact in $C([0,1], \mathcal{A}_0)$ through an Arzela-Ascoli argument. Then, in proposition 3.4, we will show that $\mathbf{A}$ has a closed graph i.e. if $Y_n \rightharpoonup Y$ in $\mathbb{L}^2_B$ and $D(\mathbf{A}(Y_n), A) \to 0$ then $\mathbf{A}(Y) = A$. □

3.1.1. *Proof of the result.*

Proposition 3.3. *Let* $r > 0$ *and* $B(0,r) = \{ \ Y \in \mathbb{L}^2_B \mid ||Y||_2 \leq r \ \}$. *Then,* $\mathbf{A}(B(0,r))$ *is relatively compact in* $C([0,1], \mathcal{A}_0), D)$.

Proof. Since $||Y||_1 \leq r$ for all $Y \in B(0,r)$, we deduce from lemma 2.4 that there exists K_1 such that for all $Y \in B(0,r)$, all $t \in [0,1]$, and all $m, m' \in M$

$$
\begin{aligned}
(32)\quad d_M(\Phi_t(Y)(m), \Phi_t(Y)(m')) + d_M(\Phi_t(Y)^{-1}(m), \Phi_t(Y)^{-1}(m')) \\
+ d_G(H_t(Y)(m), H_t(Y)(m')) \leq K_1 d_M(m, m').
\end{aligned}
$$

Now, there exists also an $R > 0$ depending only on r such that for all $Y \in B(0,r)$, all $t \in [0,1]$ and all $m \in M$ we have $d_G(1_G, H_t(Y)(m)) \leq R$. Since the metric on G is left invariant, the geodesic balls on G are compact. Hence, there exists a compact $K_0 \subset M \times G$ such that

$$
(33)\qquad \mathbf{A}_t(Y)(M) \subset K_0;\ t \in [0,1],\ Y \in B(0,r).
$$

From (32) and (33), we deduce using Arzela-Ascoli's theorem that $\mathcal{K} = \{ \ \mathbf{A}_t(Y) \mid t \in [0,1],\ Y \in B(0,r) \ \}$ is relatively compact in $(\mathcal{A}_0, d_0)$. Hence, to prove the lemma, it is sufficient (using a second time Arzela-Ascoli's theorem) to show that $\{ \ \mathbf{A}(Y) \mid Y \in$

$B(0,r)$ } is an uniformly equicontinuous family in $C([0,1],\mathcal{A}_0)$ i.e. to get a control uniform in $m\in M$, $Y\in B(0,r)$ and $|t-t'|$ of

$$d_M(\Phi_t(Y)(m),\Phi_{t'}(Y)(m))+d_M(\Phi_t^{-1}(Y)(m),\Phi_{t'}(Y)^{-1}(m)) + d_G(H_t(Y)(m),H_{t'}(Y)(m)). \tag{34}$$

Using Cauchy-Schwartz inequality and property (H2) of the norm on B, we get a $K(r)>0$ such that

$$d_M(\Phi_t(Y)(m),\Phi_{t'}(Y)(m))+d_G(H_t(Y)(m),H_{t'}(Y)(m))\le K(r)\sqrt{|t'-t|}.$$

Moreover, we get also that $d_M(\Phi_{t'}(Y)\circ\Phi_t(Y)^{-1}(m),m)\le K(r)\sqrt{|t'-t|}$ for all $0\le t\le t'\le 1$ so that using (32) we get the

$$d_M(\Phi_t^{-1}(Y)(m),\Phi_{t'}(Y)^{-1}(m))\le K(r)\sqrt{|t'-t|}.$$

Hence the uniform equicontinuity is proved and the proof is complete. □

Proposition 3.4. *Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{L}^2_B$, let $Y\in\mathbb{L}^2_B$ and $A\in\mathcal{A}_B$ such that*

$$Y_n \underset{n\to+\infty}{\rightharpoonup} Y \text{ and } D(\mathbf{A}(Y_n),A)\underset{n\to+\infty}{\to} 0.$$

Then,

$$\mathbf{A}(Y)=A.$$

The previous proposition shows that the graph of $\mathbf{A}$ is closed for the weak topology on $\mathbb{L}^2_B$ and the metric D on $C([0,1],\mathcal{A}_0)$. The idea of the proof is quite simple. For any fixed time t and any starting point $(m_0,h_0)\in M\times G$, then the integral curve associated with an element $Y\in\mathbb{L}^2_B$ is a continuous linear mapping in Y. Obviously, this is not true for an integral curve on a manifold but holds in a chart. Therefore, one can expect that the weak convergence in $\mathbb{L}^2_B$ is sufficient to identify A and $\mathbf{A}(Y)$.

The proof will be split in two preliminary lemmas on the properties in a chart of integral curves associated to the flow mapping. Let (Ω_1,f_1) (resp. (Ω_2,f_2)) be a chart from the atlas of M (resp. of G). Now, let ω_1 (resp. ω_2) be a relatively compact open set of M (resp. of G) such that $\omega_1\subset\overline{\omega_1}\subset\Omega_1$ (resp $\omega_2\subset\overline{\omega_2}\subset\Omega_2$). We assume also that there exists $R>0$ such that for all $m,m'\in\omega_1$ and all $h,h'\in\omega_2$, there exists $p_1\in C^\infty([0,1],\omega_1)$ and $p_2\in C^\infty([0,1],\omega_2)$ such that

$$d_M(m,m')\le R\int_0^1 |\frac{dp_1}{du}|du \text{ and } d_G(h,h')\le R\int_0^1|\frac{dp_2}{du}|du. \tag{35}$$

For all $\eta>0$, let

$$\mathcal{C}_\eta=\{\ \gamma=(\gamma_1,\gamma_2)\in C([0,\eta],M\times G)\mid \gamma([0,\eta])\subset\omega_1\times\omega_2\ \}.$$

Let $p = \dim(M)$ and $q = \dim(G)$. For all $0 \leq \delta \leq \eta$, all $t_0 \geq 0$ such that $t_0 + \eta \leq 1$, all $\gamma \in \mathcal{C}_\eta$, we define a linear mapping $l_\gamma^{t_0,\delta}$ from $\mathbb{L}_B^1$ to $\mathbb{R}^p \times \mathbb{R}^q$ by

$$l_\gamma^{t_0,\delta}(Y) = \Big(\int_0^\delta d_{\gamma_1(s)} f_1 \left(U(t_0 + s, \gamma_1(s)) \right) ds, \int_0^\delta d_{\gamma_2(s)} f_2(\gamma_2(s) Z(t_0 + s, \gamma_1(s))) ds \Big),$$

for all $Y = (U, Z) \in \mathbb{L}_B^1$.

Lemma 3.5. *Let $\eta_0 > 0$ and $t_0 \geq 0$ such that $t_0 + \eta_0 \leq 1$. There exists $K > 0$ such that for all $0 \leq \eta \leq \eta_0$, all $Y \in \mathbb{L}_B^1$ and all $0 \leq \delta \leq \eta$, it holds that*

(i) for all $\gamma \in \mathcal{C}_\eta$, we have

$$|l_\gamma^{t_0,\delta}(Y)|_{\mathbb{R}^p \times \mathbb{R}^q} \leq K ||Y||_1,$$

(ii) for all γ and $\gamma' \in \mathcal{C}_\eta$, we have

$$\begin{aligned} &|l_\gamma^{t_0,\delta}(Y) - l_{\gamma'}^{t_0,\delta}(Y)|_{\mathbb{R}^p \times \mathbb{R}^q} \\ &\qquad \leq K \Big(\sup_{s \leq \eta} (d_G(\gamma_2(s), \gamma_2'(s)) + d_M(\gamma_1(s), \gamma_1'(s))) \Big) \int_0^\delta |Y_{t_0+s}|_e ds. \end{aligned}$$

Proof. Since $\overline{\omega_1} \times \overline{\omega_2}$ is a compact set in $\Omega_1 \times \Omega_2$, there exists $C > 0$ such that for all $(m, h) \in \omega_1 \times \omega_2$, all $u, u' \in T_m M$ and all $w, w' \in T_h G$, we have

$$\text{(36)} \quad \begin{cases} |d_m f_1(u)|_{\mathbb{R}^p} \leq C |u|_m, & |d_h f_2(w)|_{\mathbb{R}^q} \leq C |w|_h, \\ \text{and} & \\ |d_m^2 f_1(u \otimes u')|_{\mathbb{R}^p} \leq C |u|_m |u'|_m, & |d_h^2 f_2(w \otimes w')|_{\mathbb{R}^q} \leq C |w|_h |w'|_h. \end{cases}$$

Then, for all $\gamma \in \mathcal{C}_\eta$ and all $Y \in \mathbb{L}_B^1$, we have

$$|l_\gamma^{t_0,\delta}(Y)|_{\mathbb{R}^p \times \mathbb{R}^q} \leq C \int_0^\delta (|U_s|_\infty + |Z_s|_\infty) ds.$$

Since the norm $|\ |_e$ is admissible, (i) is proved.

Now, let $Y \in \mathcal{T}^\infty$ and $\gamma, \gamma' \in \mathcal{C}_\eta$. For all $0 \leq s \leq \eta$, we deduce from (35) that there exists $p_1 \in C^\infty([0,1], \omega_1)$ and $p_2 \in C^\infty([0,1], \omega_2)$ such that $d_M(\gamma_1(s), \gamma_1'(s)) \leq R \int |\frac{dp_1}{du}|$ and $d_G(\gamma_2(s), \gamma_2'(s)) \leq R \int |\frac{dp_2}{du}|$. Moreover, we have

$$|\frac{D}{du}(U(t_0 + s, p_1))| \leq |\nabla U_s|_\infty |\frac{dp_1}{du}|,$$

and

$$|\frac{D}{du}(p_2 Z(t_0 + s, p_2))| \leq (|\nabla Z_s|_\infty + K_G |Z_s|_\infty) |\frac{dp_2}{du}|,$$

where

$$K_G = \sup\{ \ |\nabla_w^G W^G(1_G)|_{1_G} \ | \ w, W \in \mathfrak{G}, |W|_{1_G} = |w|_{1_G} = 1 \ \},$$

and W^G is the left invariant vector fields on G associated to W. Hence, using (36), we get

$$|d_{\gamma_1(s)} f_1(U(t_0+s,\gamma_1(s)) - d_{\gamma_1'(s)} f_1(U(t_0+s,\gamma_1'(s))|_{\mathbb{R}^p} \leq CR(|U_s|_\infty + |\nabla U_s|) d_M(\gamma_1(s),\gamma_1'(s)),$$

and

$$|d_{\gamma_2(s)} f_2(\gamma_2(s)Z(t_0+s,\gamma_2(s)) - d_{\gamma_2'(s)} f_2(\gamma_2'(s)Z(t_0+s,\gamma_2'(s))|_{\mathbb{R}^q} \leq CR((K_G+1)|Z_s|_\infty + |\nabla Z_s|_\infty) d_G(\gamma_2(s),\gamma_2(s')).$$

Integrating over s, we get (ii) so that the proof is complete. □

Lemma 3.6. *Let $\eta > 0$ and $t_0 \geq 0$ such that $t_0 + \eta \leq 1$. Let $\gamma \in \mathcal{C}_\eta$ and $Y \in \mathbb{L}^1_B$. Assume that there exists $m \in M$ such that $\mathbf{A}_{t_0}(Y)(m) = \gamma(0)$. Then (i) and (ii) are equivalent:*

(i) $\mathbf{A}_{t_0+u}(Y)(m) = \gamma(u)$ for all $0 \leq u \leq \eta$,
(ii) $l^{t_0,u}_\gamma(Y) = \hat{\gamma}(u) - \hat{\gamma}(0)$ where $\hat{\gamma} = (f_1 \circ \gamma_1, f_2 \circ \gamma_2)$.

Proof. $(i) \Rightarrow (ii)$: If $Y \in \mathcal{T}^\infty$, (ii) is nothing else than an integrated version of (10) in a chart. Now, if $Y \in \mathbb{L}^1_B$, let $(Y_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{T}^\infty$ such that $||Y_n - Y||_1 \to 0$. For all $n \geq 0$, we define $\gamma^n \in C([0,\eta], M \times G)$ by $\gamma^n(s) = \mathbf{A}_{t_0+s}(Y_n)(m)$. From proposition 2.10 we get that

$$\lim_{n\to\infty} D(\mathbf{A}(Y_n), \mathbf{A}(Y)) = 0. \tag{37}$$

We deduce from (37) that $\sup_{0\leq s\leq\eta} d_G(\gamma_2^n(s),\gamma_2(s)) + d_M(\gamma_1^n(s),\gamma_1(s)) \to 0$. Hence $\gamma^n \in \mathcal{C}_\eta$ for sufficiently large n. Using $(i) \Rightarrow (ii)$ for γ^n and Y_n we get

$$\lim_{n\to\infty} l^{t_0,u}_{\gamma_n}(Y_n) = \lim_{n\to\infty} \hat{\gamma}^n(u) - \hat{\gamma}^n(0) = \hat{\gamma}(u) - \hat{\gamma}(0),$$

for all $0 \leq u \leq \eta$. Moreover, from (i) and lemma 3.5, we get that

$$\lim_{n\to\infty} l^{t_0,u}_{\gamma^n}(Y^n) = l^{t_0,u}_\gamma(Y),$$

so that $(i) \Rightarrow (ii)$ is proved.

$(ii) \Rightarrow (i)$: Let $\gamma' \in C([0,\eta], M \times G)$ defined by $\gamma'(s) = \mathbf{A}_{t_0+s}(Y)(m)$. Let $s_0 = \inf\{\ s \leq \eta \mid s \geq 0,\ \gamma(s) \neq \gamma'(s)\ \}$. Assume that $s_0 < \eta$. We will prove that we get a contradiction. Changing t_0 in $t_0 + s_0$, we can assume that $s_0 = 0$. There exist $0 \leq \eta' \leq \eta$ such that $\gamma' \in \mathcal{C}_{\eta'}$. Now, let $h \in C([0,\eta'],\mathbb{R})$ be defined by

$$h(s) = |\hat{\gamma}(s) - \hat{\gamma}'(s)|_{\mathbb{R}^p\times\mathbb{R}^q},$$

where $\hat{\gamma}' = (f_1 \circ \gamma_1', f_2 \circ \gamma_2')$. We get from (ii) and $(i) \Rightarrow (ii)$ that for all $0 \leq u \leq \eta'$,

$$h(u) = |l^{t_0,u}_\gamma(Y) - l^{t_0,u}_{\gamma'}(Y)|_{\mathbb{R}^p\times\mathbb{R}^q} \leq K \int_0^u |Y_{t_0+s}|_e h(s) ds.$$

Since $h(0) = 0$, we deduce that $h(u) = 0$ for all $0 \leq u \leq \eta'$ so that $\gamma(u) = \gamma'(u)$ for all $0 \leq u \leq \eta'$. This contradicts $s_0 = 0$. $\square$

Proof. (proposition 3.4) First, notice that since $Y_n \rightharpoonup Y$, there exists $R > 0$ such that $||Y||_2 \leq R$ and $||Y_n||_2 \leq R$ for all $n \geq 0$. Now, applying proposition 3.3, we deduce that $(\mathbf{A}(Y_n))_{n\in\mathbb{N}}$ is a relatively compact sequence in $C([0,1], \mathcal{A}_0), D)$. Let $(\mathbf{A}(Y_{n_k}))_{k\geq 0}$ be any converging subsequence. There exists $A \in C([0,1], \mathcal{A}_0)$ such that $D(A, \mathbf{A}(Y_{n_k})) \to 0$. We should prove that $A = \mathbf{A}(Y)$. Let $m \in M$ and $t_0 = \inf\{\ t > 0 \mid A_t(m) \neq \mathbf{A}_t(Y)(m)\ \}$. Assume that $t_0 < 1$ and denote $(m_0, h_0) = A_{t_0}(m) = \mathbf{A}_{t_0}(Y)(m)$. We consider a chart (Ω_1, f_1) (resp. (Ω_2, f_2)) and a relatively compact open set $\omega_1 \subset \overline{\omega_1} \subset \Omega_1$ (resp. $\omega_2 \subset \overline{\omega_2} \subset \Omega_2$) such that $m_0 \in \omega_1$ (resp. $h_0 \in \omega_2$) and (35) holds. Since there exists $K > 0$ such that

$$d_M(m_0, \Phi_{t_0+u}(Y_n)(m)) + d_G(h_0, H_{t_0+u}(Y_n)(m)) \leq K\int_0^u |Y_n|_e ds \leq K\sqrt{u}R,$$

there exists $\eta > 0$ such that $\mathbf{A}_{t_0+u}(Y_n)(m) \in \omega_1 \times \omega_2$ for all $0 \leq u \leq \eta$. Hence $A_{t_0+u}(m) \in \omega_1 \times \omega_2$ for all $0 \leq u \leq \eta$. We will see that if $\gamma(s) = A_{t_0+s}(m)$ for all $0 \leq s \leq \eta$, then $l_\gamma^{t_0,u}(Y) = \hat{\gamma}(u) - \hat{\gamma}(0)$ for all $0 \leq u \leq \eta$ where $\hat{\gamma} = (f_1 \circ \gamma_1, f_2 \circ \gamma_2)$ so that we will deduce from lemma 3.6 that $A_{t_0+u}(m) = \mathbf{A}_{t_0+u}(Y)(m)$ for all $0 \leq u \leq \eta$ which contradicts $t_0 < 1$. Indeed,

$$l_\gamma^{t_0,u}(Y) = l_\gamma^{t_0,u}(Y - Y_{n_k}) + \left(l_\gamma^{t_0,u}(Y_{n_k}) - l_{\gamma^{n_k}}^{t_0,u}(Y_{n_k})\right) + l_{\gamma^{n_k}}^{t_0,\eta}(Y_{n_k})$$

where $\gamma^n(s) = \mathbf{A}_{t_0+s}(Y^n)(m)$ for all $0 \leq s \leq \eta$. Since $l_\gamma^{t_0,u}$ is continuous linear mapping from $\mathbb{L}_B^1$ to $\mathbb{R}^p \times \mathbb{R}^q$ (so also on $\mathbb{L}_B^2$), we deduce that

$$\lim_{k\to\infty} l_\gamma^{t_0,u}(Y^{n_k} - Y) = 0.$$

Moreover, using lemma 3.5, the fact that $||Y^n||_1$ is bounded, and the uniform convergence of γ^{n_k} to γ, we deduce that

$$\lim_{k\to\infty} \left(l_\gamma^{t_0,u}(Y^{n_k}) - l_{\gamma^{n_k}}^{t_0,\delta}(Y^{n_k})\right) = 0.$$

Since from lemma 3.6 we have $l_{\gamma^n}^{t_0,u}(Y^n) = \hat{\gamma^n}(u) - \hat{\gamma^n}(0)$, we get the result. $\square$

3.2. Existence of geodesics and weak compactness. In the next theorem, we state the most important consequences of the theorem 3.2. We will see that through the flow mapping at time 1 i.e. $\mathbf{a}$, we get on $\mathcal{A}_B$ the good property of the weak topology in $\mathbb{L}_B^2$. In order to emphasize this correspondence, we will call the weak topology on $\mathcal{A}_B$ the topology given by the metric d_0 and the strong one the topology given by d_B.

Theorem 3.7. *Let $\mathcal{A}_B$ be an Hilbert sub-group of $\mathcal{A}_0$.*

(i) Let $a \in \mathcal{A}_B$. Then, there exists $Y \in \mathbb{L}_B^2$ such that

$$a = \mathbf{a}(Y) \text{ and } ||Y||_2 = ||Y||_1 = d_B(e, a).$$

(ii) Let $r > 0$ and let $B(e,r) = \{ a \in \mathcal{A}_B \mid d_B(e,a) \leq R \}$ be the strong closed ball in $\mathcal{A}_B$. Then $B(e,r)$ is compact for the metric d_0.

(iii) The application $a \to d_B(e,a)$ is lower semi-continuous for the metric d_0 on $\mathcal{A}_B$.

Proof. Since (iii) is a straightforward consequence of (ii) we will only detail the proof of (i) and (ii).

(Proof of (i)) Let $K = \{ Y \in \mathbb{L}^2_B \mid \mathbf{a}(Y) = a \text{ and } ||Y||_2 \leq 2d_B(e,a) \}$. The set K is non-empty and from theorem 3.2, we get that K is a compact set for the weak topology. Then there exists a sequence $(Y_n)_{n\in\mathbb{N}}$ in K and $Y \in K$ such that $Y_n \rightharpoonup Y$ and $||Y_n||_2 \to d_B(e,a)$. Hence

$$d_B(e,a) \leq ||Y||_1 \leq ||Y||_2 \leq \liminf ||Y_n||_2 \leq d_B(e,a),$$

so that the proof of (i) is complete.

(Proof of (ii)) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $B(0,r)$. From part (i) of the theorem, we deduce that there exists a sequence $(Y_n)_{n\in\mathbb{N}}$ in $\mathbb{L}^2_B$ such that for all $n \in \mathbb{N}$ we have $\mathbf{a}(Y_n) = a_n$ and $||Y_n||_2 = d_B(e,a_n) \leq r$. Then extracting a subsequence weakly converging in $\mathbb{L}^2_B$ towards an element Y, we deduce from theorem 3.2 that there exists a subsequence of $(a_n)_{n\in\mathbb{N}}$ converging for the metric d_0 to $a = \mathbf{a}(Y)$. Moreover, one have $d_B(e,a) \leq ||Y||_1 \leq ||Y||_2 \leq r$ so that $a \in B(e,r)$ and the proof of (ii) is complete. □

The equality $||Y||_2 = ||Y||_1$ means that $|Y_s|_e$ keeps for almost every $s \in [0,1]$ a constant value so that $t \to \mathbf{A}(Y)_t$ should be interpreted as a geodesic curves in $\mathcal{A}_B$ from e to a. Using the left invariance of the metric on $\mathcal{A}_B$, we deduce that for any $a' \in \mathcal{A}_B$, $t \to a'(Y)_t$ is a geodesic curve from a' to $a'a$ so that we get from theorem 3.7 the existence of a geodesic curve between two arbitrary points in $\mathcal{A}_B$.

One should notice here that we have proved before that $\mathcal{A}_B$ is complete as a metric space for the distance d_B. We known (Hopft and Rinow theorem, see [6] p343), that for any finite dimensional Riemannian manifold, such a property implies the existence of a minimal geodesic between points. The previous theorem shows that we can get such a result in our infinite dimensional setting. This should suggest that one could define an exponential mapping Exp from the Hilbert space $(\widetilde{T}_e\mathcal{A}_B, \langle\ ,\ \rangle_e)$ (given by the completion of $\mathfrak{X}(M) \times C^\infty(M,G)$) to $\mathcal{A}_B$ which should be onto. This contrast with the fact that the exponential mapping exp on $\mathcal{A}_B$ considered as a Lie group is not generally onto. However, this is until now just a conjecture since we should established first a regularity result on the geodesic curves to map any geodesic curve with its velocity at the starting point. The reader will find in the subsection 4.1, a discussion about the role of Exp in a problem of control related to our applications to pattern recognition.

3.3. Existence of solution to some variational problems.

Theorem 3.8. *Let $\mathcal{A}_B$ be a Hilbert sub-group. Let $E : \mathcal{A}_B \longrightarrow \mathbb{R}_+$ be a continuous function for the distance d_0 on $\mathcal{A}_B$. Let $R : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a non decreasing function such that $R(x) \rightarrow +\infty$ as $x \rightarrow +\infty$. Let $W : \mathcal{A}_B \longrightarrow \mathbb{R}_+$ be defined by*

$$W(a) = E(a) + R(d_B(e, a)).$$

Then, there exists $\hat{a} \in \mathcal{A}_B$ such that

$$W(\hat{a}) = \inf_{a \in \mathcal{A}_B} W(a).$$

Proof. From theorem 3.7 (iii), we deduce that W is lower semi-continuous for the metric d_0 on $\mathcal{A}_B$. It is sufficient to prove that the level sets are compact. However, for any $\lambda > 0$, if $(a_n)_{n \in \mathbb{N}}$ is a sequence in $\mathcal{A}_B$ such that $W(a_n) \leq \lambda$, then, using the fact that E is nonnegative and the condition on the limit of R at infinity, we deduce that $d_B(e, a_n)$ is bounded. Since the strong balls are compact according to theorem 3.7 (ii) and that W is lower semi-continuous, there exists a subsequence $(a_{n_k})_{k \geq 0}$ and $a \in \mathcal{A}_B$ such that $d_0(a_{n_k}, a) \rightarrow 0$ and $W(a) \leq \liminf A(a_{n_k}) \leq \lambda$. □

This last theorem gives a general existence result of a minimizer on the group $\mathcal{A}_B$ for a large class on variational problem. We will see below (corollary 3.9) that the theorem covers the problem of pattern matching given in the introduction under a weak condition on the function L arising in the "external" energy part (as usual called in the active contours framework).

Corollary 3.9. *Let $\mathcal{A}_B$ be an Hilbert sub-group of $\mathcal{A}$. Assume that the action on X $(g, x) \rightarrow gx$ is continuous and let f be a continuous pattern i.e. $f \in C(M, X)$. Let $L \in C(X \times X, \mathbb{R})$ and $\widetilde{f} \in \mathcal{P}$ such that $\widetilde{f}(M)$ is relatively compact. Now, consider the function W on $\mathcal{A}_B$ defined by*

$$W(a) = \int_M L(\widetilde{f}, af) d\mu + \frac{1}{2} d_B(e, a)^2. \tag{38}$$

Then, there exists $\hat{a} \in \mathcal{A}_B$ such that

$$W(\hat{a}) = \inf_{a \in \mathcal{A}_B} W(a).$$

Proof. We deduce from theorem 3.8 that it is sufficient to prove that the function $a \rightarrow \int_M L(\widetilde{f}, af) d\mu$ is continuous for the distance d_B on $\mathcal{A}_B$. Since f is continuous and M is compact, $f(M)$ is a compact subset of X. Moreover, let $\mathcal{B}$ be a bounded subset of $\mathcal{A}_B$ for the distance d_B. We deduce that $\mathcal{B}_G = \{ \ h(m) \mid m \in M, \ a = (h, \phi) \in \mathcal{B} \ \}$ is bounded in G. Since G is complete, $\mathcal{B}_G$ is relatively compact in G. Now, using the continuity of the action, we get that the set $\mathcal{B}_X = \{ \ gx \mid g \in \mathcal{B}_G \ x \in f(M) \ \}$ is relatively compact in X. Moreover, for all $a \in \mathcal{B}$ and all $m \in M$, $(af)(m) \in \mathcal{B}_X$. Hence, since $\widetilde{f}(M)$ is relatively compact and $L \in C(X \times X, \mathbb{R})$, we deduce that $\mathcal{B}_{\mathbb{R}} = \{ L(\widetilde{f}(m), (af)(m)) \mid m \in M \}$ is relatively compact in $\mathbb{R}$ so that $\mathcal{B}_{\mathbb{R}}$ is bounded. Finally, since for each fixed $m \in M$, $a \to L(\widetilde{f}(m), (af)(m))$ is continuous for the distance d_B, for any $a \in \mathcal{A}_B$ and any sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathcal{A}_B$ such that $d_B(a, a_n) \to 0$, considering $\mathcal{B} = \{ a_n \mid n \geq 0\} \cup \{a\}$, we deduce from the dominated convergence theorem that

$$\int_M L(\widetilde{f}, a_n f) d\mu \to \int_M L(\widetilde{f}, af) d\mu,$$

so that the result is proved. $\square$

3.4. Gradient descent. For Hilbert sub-groups $\mathcal{A}_B$, the tangent spaces are separable Hilbert spaces so that for any differentiable application $E : \mathcal{A}_B \to \mathbb{R}$, one can define the gradient of E.

Definition 3.10. Let $\mathcal{A}_B$ be a Hilbert sub-group of $\mathcal{A}_0$ and let $E : \mathcal{A}_B \to \mathbb{R}$ be a differentiable application. Then for any $a \in \mathcal{A}_B$, we denote $\nabla_a E$ the unique element of $\widetilde{T}_a \mathcal{A}_B$ such that for all $y \in \widetilde{T}_a \mathcal{A}_B$ we have $d_a E(y) = \langle \nabla_a E, y \rangle_a$.

For pattern classification and recognition tasks, we will have to consider non linear evolution equations on $\mathcal{A}_B$ defined by $\frac{da}{dt} = -\nabla_a E$. More generally, we have to look for integrability conditions of vector fields on $\mathcal{A}_B$. We come back to our important examples of differentiable applications introduced in theorem 2.17.

Proposition 3.11. *Assume that $\mathcal{A}_B$ is a Hilbert sub-group of $\mathcal{A}_0$ and that the action $(g, x) \to gx$ is C^2. Assume that $E : \mathcal{A}_B \to \mathbb{R}$ is defined as in theorem* 2.17 *(i) or (ii). Then the gradient field ∇E is bounded and strongly Lipschitz.*

Proof. Note that it is sufficient to prove the result for $E : \mathcal{A}_B \to \mathbb{R}$ defined as in theorem 2.17 (ii). We use the notation of the proof of the theorem 2.17. Let $a \in \mathcal{A}_B$ and $y \in \widetilde{T}_a \mathcal{A}_B$. Since

$$|d_a E(y)| \leq |\int_M \langle \nabla^R l(\widetilde{f}, a), y \rangle^R d\mu| \leq \sup_{x\in X} |\nabla^R l_x|_\infty |y|_a, \tag{39}$$

we deduce that $|\nabla_a E|_a \leq \sup_{x\in X} |\nabla^R l_x|_\infty$ so that ∇E is bounded. We want to prove now that ∇E is strongly Lipschitz. Let a and a' be in $\mathcal{A}_B$, let $m \in M$, let $y \in \widetilde{T}_e \mathcal{A}_B$ and let $p \in C^\infty([0,1], M \times G)$ such that $p(0) = a(m)$ and $p(1) = a'(m)$. Let $q \in C^1([0,1], \mathbb{R})$ be defined by $q(s) = \langle \nabla^R l(\widetilde{f}(m), p(s)), p(s)y \rangle^R$ where $p(s)y$ denotes $(u(p_1(s)), p_2(s) z(p_1(s)))$ with $p = (p_1, p_2)$ and $y = (u, z)$. One computes

$$\frac{dq}{ds} = \langle \nabla^R_{\frac{dp}{ds}} \nabla^R l(\widetilde{f}(m), p(s)), p(s)y \rangle^R + \langle \nabla^R l(\widetilde{f}(m), p(s)), \frac{D^R}{ds}(p(s)y) \rangle^R.$$

Since $|\frac{D^R}{ds}(p(s)y)| = |(\nabla^M_{\frac{dp_1}{ds}}u, p_2(s)(\nabla^{\mathfrak{G}}_{\frac{dp_1}{ds}}z + \nabla^G_{p_2^{-1}\frac{dp_2}{ds}}z^G))| \leq K|y|_e|\frac{dp}{ds}|$, we get $|\frac{dq}{ds}| \leq M|y|_e|\frac{dp}{ds}|$, so that

$$|\langle\nabla^R l(\widetilde{f}(m), a(m)), (ay)(m)\rangle^R - \langle\nabla^R l(\widetilde{f}(m), a'(m)), (a'y)(m)\rangle^R| \leq M|y|_e d_0(a, a'),$$

where $M = K\sup_{x\in X}(|\nabla^R l_x|_\infty + |\nabla^R\nabla^R l_x|_\infty)$. After integration under μ, we obtain finally that for all $y \in \widetilde{T}_e\mathcal{A}_B$ we have $|d_aE(ay) - d_{a'}E(a'y)| \leq M|y|_e d_0(a, a')$ so that $|a^{-1}\nabla_a E - (a')^{-1}\nabla - a'E|_e \leq M|y|_e d_0(a, a')$. Thus, ∇E is strongly Lipschitz and the proof is complete. □

4. Application to pattern recognition

We turn back to the problem of pattern classification and matching as set in the introduction. Let $\mathcal{A}_B$ be a Hilbert sub-group of $\mathcal{A}_0$ and assume that the action is C^2. Let $L \in C^2(X \times X, \mathbb{R})$ be non negative. The value of $L(x, x')$ should be interpreted as a distance between x and x'. Let $(f_i)_{1\leq i\leq p}$ be a family of C^2 patterns in $\mathcal{P}$ called the template patterns. Now, let $\widetilde{f} \in \mathcal{P}$ be the observed pattern. For all $i \in \{1, \cdots, p\}$, we define $E_i : \mathcal{A}_B \to \mathbb{R}$ by and $W_i : \mathcal{A}_B \to \mathbb{R}$ by

$$E_i(a) = \int_M L(\widetilde{f}, af_i)d\mu \text{ and } W_i(a) = E_i(a) + \frac{1}{2}d_B(e, a)^2.$$

Let $S_i = \inf_{a\in\mathcal{A}_B} W_i(a)$ be called the *score* of the template f_i. We will say that $\widetilde{f}$ belongs to the class of f_i if $S_i \leq S_j$ for all $j \neq i$. The problem of classification and the problem of pattern matching is reduced to the computation of a minimum $\hat{a}_i \in \mathcal{A}_B$ such that $W_i(\hat{a}_i) = \inf_{a\in\mathcal{A}_B} W_i(a)$. The existence of $\hat{a}_i$ under a weak set of hypothesis on L, $\widetilde{f}$ and f_i is a consequence of corollary 3.9 if we assume the $\mathcal{A}_B$ is an Hilbert sub-group of $\mathcal{A}_0$. This existence result is particularly welcome since the lack of existence result is one of the main drawback of most of the variational approach in pattern analysis. Now, on the computation side, we still need a numerical scheme to get $\hat{a}_i$.

4.1. Optimal control approach. From now, we focus on the matching problem for a given template f and a given observed pattern $\widetilde{f}$ so that we will forget the under-script i. A first approach is to reformulate the problem of the computation of $\hat{a}$ as a problem of optimal control. Indeed, let $H([0,1], \mathcal{A}_B) = \mathbf{A}(\mathbb{L}^2_B)$. Now, let $J : H([0,1], \mathcal{A}_B) \to \mathbb{R}$ be defined by

$$J(A) = E(A_1) + \frac{1}{2}\int_0^1 |\frac{dA_s}{ds}|^2_{A_s} ds,$$

where for any $A = \mathbf{A}(Y)$, $\frac{dA_s}{ds}$ denotes A_sY_s and $|\frac{dA_s}{ds}|_{A_s}$ denotes $|A_sY_s|_{A_s} = |Y_s|_e$. A minimum $\hat{A}$ of J is a solution of an optimal control problem. The existence of such an $\hat{A}$ in $H([0,1], \mathcal{A}_B)$ is again a consequence of theorem 3.2 since we deduce easily from it as in corollary 3.9 that $J \circ \mathbf{A}$ is lower semi-continuous on $\mathbb{L}^2_B$ with compact level sets. Obviously, $\hat{A}_1$ is a minimum of W. Now, working formally, one could write the solution of the associated Hamilton Jacobi equation as (see [5])

$$\frac{\partial V}{\partial t} + \frac{1}{2}|\frac{\partial V}{\partial a}|^2_a = 0;\ V(1,a) = E(a), \tag{40}$$

where the Hamiltonian is given by $H(t,a,y) = \frac{1}{2}|y|^2_a$. Then, $\hat{A}$ is a solution of

$$\frac{d\hat{A}_t}{dt} = -\frac{\partial V}{\partial a}(t, \hat{A}_t). \tag{41}$$

Of course, the existence and the regularity of a solution of (40) in our infinite dimensional group situation cannot be stated rigorously without more information on the differential structure of $\mathcal{A}_B$. From a geometrical point of view, assume that for each $a \in \mathcal{A}_B$ and each $y \in \widetilde{T}_a\mathcal{A}_B$ we can define for all $t \in \mathbb{R}_+$ the geodesic curve $t \to \mathrm{Exp}(ty)$ starting from $a \in \mathcal{A}_B$ with initial velocity $y \in \widetilde{T}_a\mathcal{A}_B$. The existence of the Riemannian exponential mapping is not established since we have only proved the existence of a geodesic between points in $\mathcal{A}_B$. However, still working formally, let Ψ be the flow on $\mathcal{A}_B$ defined by

$$\Psi_t(a) = \mathrm{Exp}(t\nabla_a E),$$

(Ψ is the geodesic flow on $\mathcal{A}_B$ with initial velocity field ∇E). According to the usual theory of optimal control, if for any $t \in]0,1]$, $\hat{a}_t$ is a minimum of W_t defined by

$$W_t(a) = E(a) + \frac{1}{2t} d_B(e,a)^2,$$

we have $\Psi_t(\hat{a}_t) = e$. Hence, one can try to solve the equation below with starting point e:

$$\frac{d\hat{a}_t}{dt} = \frac{\partial(\Psi_t)^{-1}}{\partial t}(e). \tag{42}$$

Equation (42) is more tractable than equation (41) since, as far as Ψ_t is invertible, $\frac{\partial(\Psi_t)^{-1}}{\partial t}(e)$ depends only on Ψ_t, $\frac{\partial \psi_t}{\partial t}$ and the spatial derivative of Ψ at point $\hat{a}_t$. However, the non inversibility of Ψ_t occurs on the caustics (see [3] p458) which are unavoidable for a general function E if we do not restrict to small values of t. Moreover, we are far from being able to compute Ψ even locally. Hence, even formally, the resolution of the optimal control problem in our infinite dimensional situation leads to huge geometrical difficulties out of the scope of the general setting of this paper.

4.2. Sub-optimal solutions. However, if we look in the family of sub-optimal solutions, we can derive a useful algorithm which is tractable from the numerical point of view (see [18]). Indeed, consider the function $\widetilde{J} : [0,1] \times H([0,1], \mathcal{A}_B)$ defined by

$$\widetilde{J}(t, A) = E(A_t) + \frac{1}{2}\int_0^t |\frac{dA_s}{ds}|^2_{A_s} ds.$$

For a fixed A, we have

$$\frac{\partial \widetilde{J}}{\partial t}(t, A) = Q_{A_t}(\frac{dA_t}{dt}),$$

where $Q_a : \widetilde{T}_a\mathcal{A}_B \to \mathbb{R}$ is defined by

$$Q_a(y) = \langle \nabla_a E, y\rangle_a + \frac{1}{2}\langle y, y\rangle_a.$$

for all $a \in \mathcal{A}_B$. The minimum of Q_a is easily computed and we get

$$Q_a(-\nabla_a E) = \inf_{y\in \widetilde{T}_a\mathcal{A}_B} Q_a(y) = -\frac{1}{2}|\nabla_a E|_a^2 \leq 0,$$

so that if we consider now the solution $\widetilde{A} \in C([0,1], \mathcal{A}_B)$ of the gradient equation

$$\frac{d\widetilde{A}_t}{dt} = -\nabla_{\widetilde{A}_t} E$$

which exists thanks to theorem 2.19 and proposition 3.11. Obviously, $\widetilde{J}(t, \widetilde{A})$ is non increasing as a function of t since we have

$$\frac{\partial \widetilde{J}}{\partial t}(t, \widetilde{A}) = \inf_{y\in \widetilde{T}_{A_t}\mathcal{A}_B} Q_{\widetilde{A}_t}(y) \leq 0.$$

Now, using the inequality $(\int_0^t |\frac{dA_s}{ds}|_{A_s} ds)^2 \leq \int_0^t |\frac{dA_s}{ds}|^2_{A_s} ds$ we get that $\widetilde{J}(t, A) \leq W(A_t)$ for all $t \in [0,1]$.

From this discussion, we can now propose our sub-optimal algorithm simply based on a gradient on the function E. More precisely, assume that $\mathcal{A}_B$ is an Hilbert sub-group of $\mathcal{A}_0$ and let $E : \mathcal{A}_B \to \mathbb{R}$ be defined by

$$E(a) = \int_M L(\widetilde{f}, af)d\mu, \tag{43}$$

and satisfying the assumption of theorem 2.17 (ii). Then, we get from theorem 2.19 and 3.11 that there exists $p \in C([0,1], \mathcal{A}_B)$ which is the solution of the formal gradient equation

$$\frac{dp}{dt} = -\nabla_p E, \tag{44}$$

i.e. $p = \mathbf{A}(Y)$ with $Y_t = -p(t)^{-1}\nabla_{p(t)}E$. In fact, the gradient equation has a solution in $[0, +\infty[$. We just have to define by induction on $n \in \mathbb{N}$: $p(t+n) =$

$p(n)s_n(t)$; $t \in [0,1]$, where s_n is solution of $\frac{ds_n}{dt} = -\nabla_{s_n}(E \circ L_{p(n)})$, and $L_{p(n)}$ denotes the left multiplication by s_n. Now, let $q \in C(\mathbb{R}_+, \mathbb{R}_+)$ be defined by

$$q(t) = E(p(t)) + \frac{1}{2}\left(\int_0^t |\nabla_{p(s)}E|_e ds\right)^2 . \tag{45}$$

The function q is in fact in $C^1([0,+\infty[,\mathbb{R})$ as shown in the following proposition.

Theorem 4.1. *Let E be defined by (43) and satisfying the assumption of theorem 2.17 (ii). Let $p \in C(\mathbb{R}_+, \mathcal{A}_B)$ be the gradient descent along E defined by (44) and let $q \in C([\mathbb{R}_+, \mathbb{R}_+)$ be defined by (45). Then,*

(i) the function $q \in C^1([0,+\infty[,\mathbb{R})$ and

$$\frac{dq}{dt} = -|\nabla_p E|_p^2 + (\int_0^t |\nabla_p E|_p ds)|\nabla_p E|_p.$$

(ii) there exists $\hat{t} \in [0,+\infty[$ so that $q(\hat{t}) = \inf_{t\geq 0} q(t)$.

Proof. We start with the proof of (i). It is sufficient to prove that $c(t) = E(p(t))$ is C^1 for $t \in [0,1]$ and that $\frac{dc}{dt} = -|\nabla_{p+i}E|_p^2$. Let $(Y^n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{T}^\infty$ such that $||Y^n - Y||_1 \to 0$ where $Y_t = p(t)^{-1}\nabla_{p(t)}E$. Let $p^n = A(Y^n)$ and $c_n(t) = E(p^n(t))$. Since $Y^n \in \mathcal{T}^\infty$, one easily gets that

$$c_n(t) - c_n(0) = -\int_0^t \langle \nabla_{p^n} E,\, p^n Y^n\rangle_{p^n} ds = -\int_0^t \langle \nabla_p E,\, \nabla_p E\rangle_p ds + \epsilon_n,$$

where $\epsilon_n = \int_0^t \langle \nabla_p E,\, \nabla_p E\rangle_p ds - \int_0^t \langle \nabla_{p^n} E,\, p^n Y^n\rangle_{p^n} ds$. However,

$$\begin{aligned}
&|\langle \nabla_p E,\, \nabla_p E\rangle_p - \langle \nabla_{p^n} E,\, p^n Y^n\rangle_{p^n}| \\
&\quad \leq |\langle \nabla_{p^n} E,\, p^n Y^n - p^n p^{-1}\nabla_p E\rangle_{p^n} - \langle p^n p^{-1}\nabla_p E - \nabla_{p^n} E,\, p^n p^{-1}\nabla_p E\rangle_{p^n}| \\
&\qquad \leq |\nabla_{p^n} E|_{p^n} |Y^n - p^{-1}\nabla_p E|_e + |\nabla_p E|_p |p^{-1}\nabla_p E - (p^n)^{-1}\nabla_{p^n} E|_e.
\end{aligned}$$

Since ∇E is bounded and strongly Lipschitz, there exists $K > 0$ such that

$$|\epsilon_n| \leq K||Y^n - Y||_1 + KD(p, p^n),$$

where D is defined in definition 2.3 so that $\epsilon_n \to 0$ when n tends to infinity. Moreover, $c_n(t) - c_n(0) \to c(t) - c(0)$ so that $c(t) - c(0) = -\int_0^t |\nabla_p E|_p^2$, and the proof (i) is complete.

Concerning now the proof of (ii), we will proceed by contradiction. Assume that there exists an increasing sequence $(t_n)_{n\in\mathbb{N}}$ such that $\lim t_n = +\infty$ and $q(t_n) > q(t_{n+1})$. Then, for all $n \geq 0$, there exits $t_n^* \in]t_n, t_{n+1}[$ such that $\frac{dq}{dt}(t_n^*) < 0$ i.e. $|\nabla_{p(t_n^*)}E| \geq \int_0^{t_n^*} |\nabla_p E| ds$. Assume that $\int_0^\infty |\nabla_p E| ds > 0$. Then considering eventually a subsequence of $(t_n)_{n\in\mathbb{N}}$, we can assume that there exists $\alpha > 0$ such that $2\int_0^{t_n^*} |\nabla_p E| ds > \alpha$ for all $n \geq 0$. Since ∇E is bounded, there exists $K > 0$ (cf lemma 2.5) such that for all $t, t' \leq 0$, $d_0(p(t), p(t')) \leq K|t - t'| \exp(K(|t - t'|)$. Now, using the fact that ∇E is strongly Lipschitz, we get that there exists $\eta > 0$ such that for all $t \in [t_n^* - \eta, t_n^* + \eta]$, $|\nabla_{p(t_n^*)} E| \geq \alpha_2$. Thus, $\int_0^\infty |\nabla_p E|^2 ds = +\infty$. Since we have proved that $\frac{d}{dt}(E \circ p) = -|\nabla_p E|^2$ we deduce that $E(p(t)) \to -\infty$ which is in contradiction with the fact that E is non negative. Hence $\int_0^\infty |\nabla_p E|^2 ds = 0$. However, we get in this case that $p(t) = e$ for all $t \geq 0$ so that q is constant. This is again in contradiction with $q(t_n) > q(t_{n+1})$. The proof is complete. □

From the last theorem, we can define our sub-optimal solution of the matching problem for the observation $\widetilde{f}$ and the template f as $\widetilde{a} = p(\widetilde{t})$. Moreover, we will call $\widetilde{S} = q(\widetilde{t}) \geq W(\widetilde{a})$ the sub-optimal score of the template f. Now, if we have p templates $\{ f_i \mid 1 \leq i \leq p \}$, we define similarly $\widetilde{a}_i$ and $\widetilde{S}_i$ so that $\widetilde{i} = \text{argmin} \widetilde{S}_k$ will be called the sub-optimal solution of the classification problem and $\widetilde{a}_{\widetilde{i}}$ the sub-optimal solution of the matching problem for the family $\{ f_i \mid 1 \leq i \leq p \}$. The essential fact here, is that the sub-optimal solution of the classification problem as well as the sub-optimal solution of the matching problem are well defined and can be numerically computed, so that this approach seems very attractive.

4.3. Examples. In this subsection, we will present some applications of the previous scheme to various tasks in images and signals processing.

4.3.1. *Structural restoration of grey level images.* In this framework, we say that a grey level image is a measurable function from $M = \mathbb{R}^2/\mathbb{Z}^2$ to $X = \mathbb{R}$. One could choose for M the unit square $[0,1]^2$ but we prefer the choice of the 2-dimensional torus to have a compact manifold without boundary. This choice also allows to define translated images $f_u(m) = f(m+u)$. We single out a C^2 template in $\mathcal{P}$ denoted f and we consider an observed images $\widetilde{f} \in \mathcal{P}$. The problem of structural restoration of images as defined in [1] is described in the following way. We consider a Hilbert space of vector fields on M and we define the solution of the structural restoration problem by

$$\hat{u} = \text{argmin} \int_M (\widetilde{f}(m) - f(m + u(m))^2 d\mu + \frac{1}{2}\langle u, u \rangle_\Theta. \tag{46}$$

Given $\hat{u}$, we have a complete matching between the points of f and those of $\widetilde{f}$ by $m \to m + u(m)$ (note that $m + u(m)$ should be interpreted as a sum mod 1). This approach has been performed in the case of X-rays images of hands in [1]. However, one of the main drawbacks of this approach is that the matching $m \to m + u(m)$ is not onto nor injective on M. This problem is particularly visible when large deformations are involved.

In our framework, the problem can be well-posed. It corresponds to the case G reduced to $\{1_G\}$. Then the tangent space $\mathfrak{A}_\infty = \widetilde{T}_e \mathcal{A}_\infty$ is isomorphic to $\mathfrak{X}(M)$. Now, if we define the norm $|\ |_e$ on $\mathfrak{A}_\infty$ by

$$|y|_e = (\langle y, y \rangle_\Theta)^{1/2}, \tag{47}$$

and assuming that $|\ |_e$ is admissible (which is the case for the scalar product considered in [1]), we can define the Hilbert sub-group $\mathcal{A}_B$ of $\mathcal{A}_0$ whose tangent space $\widetilde{T}_e\mathcal{A}_B$ is isomorphic to Θ. Then, it appears that

$$\int_M (\widetilde{f}(m) - f(m+u(m))^2 d\mu + \frac{1}{2}\langle u,\, u\rangle_\Theta \tag{48}$$

is an approximation near $e = \mathrm{Id}_M$ $(u(m) = \phi(m) - m)$ to

$$\int_M (\widetilde{f}(m) - f(\phi(m))^2 d\mu + \frac{1}{2} d_B(e,\phi)^2. \tag{49}$$

Now, since $(x,x')\to(x-x')^2$ is C^2 and since G is compact, we can define the sub-optimal solution $\hat{\phi}$ of the recognition problem (in this case, we have only one template). Since $\hat{\phi} \in \mathrm{Aut}(M)$, the matching is invertible.

With our framework, we can also allows a simultaneous displacement of the points of M and a variation of the grey levels. It is sufficient to consider $G = \mathbb{R}$ with the action $gx = g + x$. In this case, we have $\mathfrak{G} = \mathbb{R}$ and $\widetilde{T}_e\mathcal{A}_\infty$ is isomorphic to $C^\infty(M, \mathbb{R}^2 \times \mathbb{R})$. The admissible norm can be given for $y = (u,z) \in \widetilde{T}_e\mathcal{A}_\infty$ by

$$|y|_e = (\langle u,\, u\rangle_\Theta + \langle z,\, z\rangle_{\Theta'})^{1/2}, \tag{50}$$

where $\langle\ ,\ \rangle_{\Theta'}$ is a scalar product on $C^\infty(M,\mathbb{R})$ such that

$$|z|_\infty + |\nabla z|_\infty \le K\langle z,\, z\rangle_{\Theta'}{}^{1/2}. \tag{51}$$

4.3.2. *Structural restoration of displacement fields.* We consider here that $X = \mathbb{R}^2$, that is the patterns are vectors fields in M. Then one can choose for G either $G = \{1_G\}$ or $\mathbb{R}^2$ according to the fact that we want or not to deform the values of $f(m)$ for $m \in M$. A more unusual case is $X = S^1$, that is $f(m)$ is an unit vector, and $G = S^1$ with the action given by the complex product (here we consider S^1 as the set of the complex numbers with norm 1). Again in this case, $\widetilde{T}_e\mathcal{A}_\infty$ is isomorphic to $C^\infty(M, \mathbb{R}\times\mathbb{R})$ and we can define the norm n by (50). For the function L, we can choose

$$L(x,x') = |x - x'|^2, \tag{52}$$

where x and x' are again considered as elements of $\mathbb{C}$ and $|x|$ denotes the usual norm on $\mathbb{C}$. One verifies easily that L is C^∞. Since G is compact, the condition for the differentiability of $E(a) = \int_M L(\widetilde{f}, af)d\mu$ is fulfilled and we can define the sub-optimal solution $\hat{p}$ of the matching problem.

4.3.3. *Active contours.* We consider here closed curves living in $\mathbb{R}^p$, so that $M = \mathbb{R}/\mathbb{Z}$ and $X = \mathbb{R}^p$. For G, a natural choice is $\mathbb{R}^p$ with the action $gx = g + x$. Given an admissible norm on $\widetilde{T}_e\mathcal{A}_\infty$ and a penalty function $L \in C^2(X, \mathbb{R}_+)$, the solution p of the formal gradient equation

$$\frac{dp}{dt} = -\nabla_p E \tag{53}$$

can be interpreted as a method of deformable contours as introduced in [13]. However, in our framework, the solution of (53) is well defined for all $t \geq 0$.

4.4. Choices of $|\ |_e$**.** We will not go further in our examples, since the framework is sufficiently general to be applied in many situations. We want here to show that the condition of admissibility on $|\ |_e$ is weak. We will consider the case when $M = \mathbb{R}^p/\mathbb{Z}^p$ and the Lie algebra of G is isomorphic to $\mathbb{R}^q$. Then, $\widetilde{T}_e\mathcal{A}_\infty$ is isomorphic to $C^\infty(M, \mathbb{R}^p \times \mathbb{R}^q)$. Let $y = (u, z) \in \widetilde{T}_e\mathcal{A}_\infty$ where u is the component on $\mathbb{R}^p$ and z on $\mathbb{R}^q$. Since M is the p-dimensional torus, one can define for all p-uplet $\hat{n} \in \mathbb{N}^p$ the Fourier coefficient $u^k_{\hat{n}}$ (resp. $z^k_{\hat{n}}$) of the k-th component of u (resp. z). Then let $(a_{\hat{n}})_{\hat{n}\in\mathbb{N}^p}$ be a sequence of non negative numbers and define

$$|y|_e = \left(\sum_{\hat{n}\in\mathbb{N}^p} a_{\hat{n}}(|u_{\hat{n}}|^2 + |z_{\hat{n}}|^2)\right)^{1/2}. \tag{54}$$

We get from the Sobolev imbeddings that $|\ |_e$ is an admissible norm if there exist $\beta \geq \alpha \geq p+3$, $K > 0$ and $K' > 0$ such that for all $\hat{n} \in \mathbb{N}^p$

$$K'(|\hat{n}| + 1)^\beta \geq a_{\hat{n}} \geq K(|\hat{n}| + 1)^\alpha \tag{55}$$

where $|\hat{n}| = \sup |n|$. For example, in the case of curves in $\mathbb{R}^2$, i.e. $M = \mathbb{R}/\mathbb{Z}$ and $X = \mathbb{R}^2$, we can choose the norm

$$|y|_e^2 = \int (\Delta u)^2 + u^2 d\mu + \int |\Delta z|^2 + |z|^2 d\mu \tag{56}$$

where Δ is the Laplacian. The case of norm $|\ |_e$ defined with the Fourier coefficients is particularly appealing for numerical reasons since one can use Fast Fourier Transform on computer in the implementation. However, the norm should be chosen according to the regularity expected on the elements of $\mathcal{A}_B$.

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